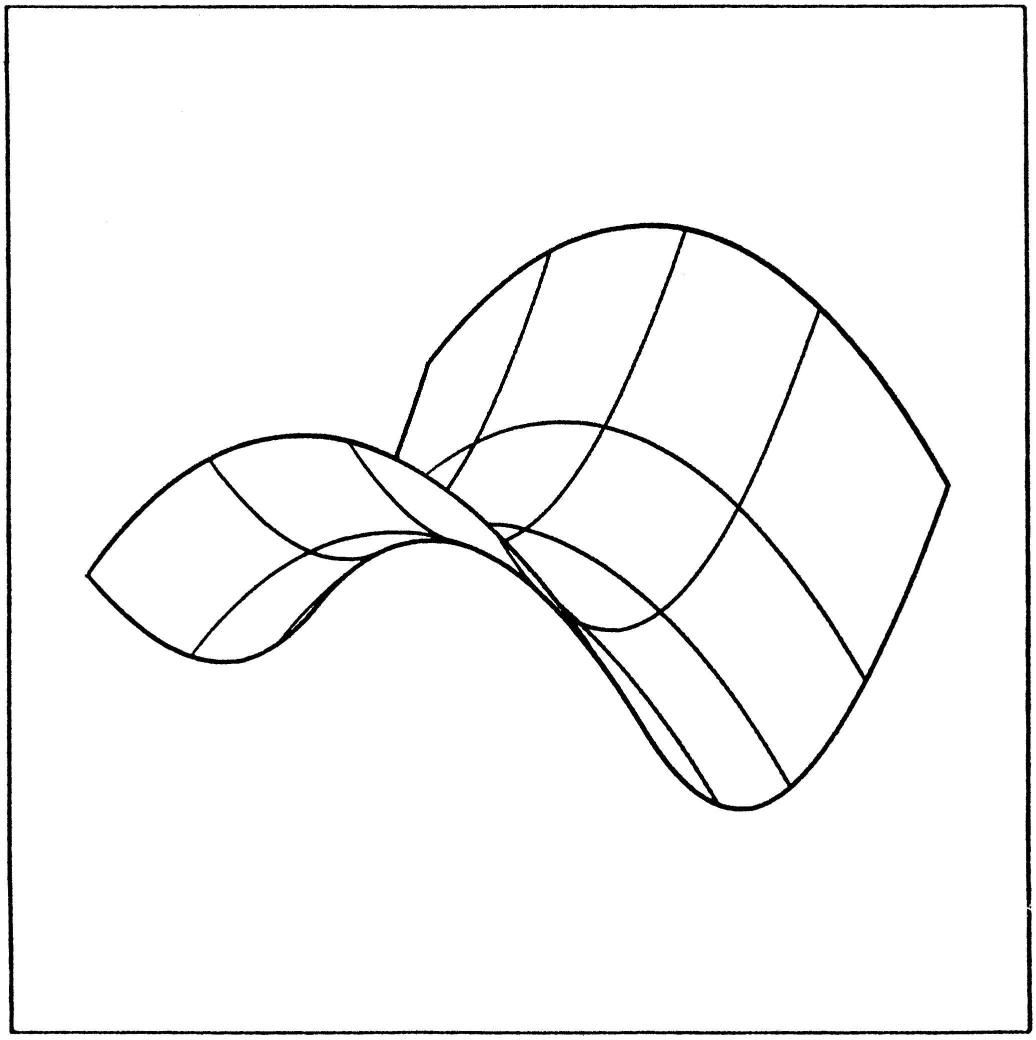
Vol. 63, No. 1 February 1990

MATHEMATICS MAGAZINE



- Second Strong Law of Small Numbers
- The River Crossing Problem
- Wronskian Harmony

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AUTHOR

Richard K. Guy taught in Britain, Singapore, India, and Canada from 1939 to 1982. He will appear in a forthcoming volume of *Mathematical People*. His interests are in number theory and combinatorics, including combinatorial games. Since 1971 he's edited the Unsolved Problems section of the *Monthly*. He also edited *Reviews in Number Theory*, 1973–1983 for the A.M.S., on whose Council he now serves. He wrote *Unsolved Problems in Number Theory* (Springer, 1981) and, with Elwyn Berlekamp & John Conway, *Winning Ways* (Academic Press). *Fair Game* (COMAP, 1989) has appeared, and he and Conway are producing *The Book of Numbers* for Scientific American Library. He likes being in the mountains. The precursor of this article won a Lester R. Ford award.

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ARTICLES

The Second Strong Law of Small Numbers

RICHARD K. GUY
The University of Calgary
Alberta, Canada T2N 1N4

You have probably already met The Strong Law of Small Numbers, either formally [**15**, **21**, **22**]

There aren't enough small numbers to meet the many demands made of them

or in some frustrated and semiconscious formulation that occurred to you in the rough-and-tumble of everyday mathematical enquiry. It is the constant enemy of mathematical discovery: at once the Scylla, shattering sensible statement with spurious exceptions, and the Charybdis of capricious coincidences, causing careless conjectures: the dilemma to search for proof or for counterexample. It fooled Fermat (Example 1 of [**21**]) and we'll meet Euler's memorable example at the end of the article.

It's time to introduce The Second Strong Law of Small Numbers:

When two numbers look equal, it ain't necessarily so!

"How can this possibly be?" I hear you ask. By way of answer I invite you to examine

Example 36 Evaluate the polynomial $(n^4 - 6n^3 + 24n^2 - 18n + 24)/24$ for $n = 1, 2, 3, \ldots$.

Examples 1 to 35 are in [**21**]; there follow forty-four more. In each, you are invited to guess what pattern of numbers is emerging, and to decide whether the pattern will persist. Many of the examples are fraudulent, but some genuine theorems are mingled in, to keep you on your toes, and there may even be an unsolved problem or two.

Examples 37 to 40 involve Pascal's triangle.

Example 37

Pascal's triangle (modulo 2) has been a perennial topic. But have you tried reading the rows as binary numbers? 1, 3, 5, 15, 17, 51, 85, 255, 257, 771, 1285, 3855, 4369, 13107, 21845, 65535, 65537, ... Remember that there are zeros outside the triangle as well, so you can also include their doubles, 2, 6, 10, 30, 34, 102, ..., their quadruples, 4, 12, 20, 60, 68, ..., and so on, as well, if you like. Do you recognize these numbers?

```
                              1
                             1 1
                            1 0 1
                           1 1 1 1
                          1 0 0 0 1
                         1 1 0 0 1 1
                        1 0 1 0 1 0 1
                       1 1 1 1 1 1 1 1
                      1 0 0 0 0 0 0 0 1
                     1 1 0 0 0 0 0 0 1 1
                    1 0 1 0 0 0 0 0 1 0 1
                   1 1 1 1 0 0 0 0 1 1 1 1
                  1 0 0 0 1 0 0 0 1 0 0 0 1
                 1 1 0 0 1 1 0 0 1 1 0 0 1 1
                1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
               1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
              1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
             1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
            1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1
           1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
          1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1
         1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1
        1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1
       1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1
      1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1
     1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1
    1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1
   1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1
  1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

Example 38 Here we've drawn Pascal's triangle with each row starting off two places to the right of the previous start, i.e. with $\binom{n}{r}$ in row n and column $2n+r$.

0	1	**2**	**3**	4	**5**	6	**7**	8	9	10	**11**	12	**13**	14	15	16	**17**	18	**19**	20	21	22	**23**	24	
0	1																								
1			**1**	**1**																					
2					1	**2**	1																		
3							1	**3**	**3**	1															
4									1	**4**	6	**4**	1												
5											1	**5**	**10**	**10**	**5**	1									
6													1	6	15	20	15	6	1						
7															1	**7**	**21**	**35**	**35**	**21**	**7**	1			
8																1	**8**	28	**56**	70	**56**	28	**8**	1	
9																			1	**9**	**36**	84	**126**	**126**	84
10																					1	**10**	45	**120**	**210**
11																							1	**11**	**55**

We've printed an entry in **bold** if it's divisible by its row number, and we've printed a column head in **bold** just if *all* the entries in the column are bold. What are these bold column heads?

Example 39 We've drawn Pascal's triangle again, but in contrast to the previous example, we've put an entry in **bold** just if it's not squarefree, i.e., just if it contains a square factor greater than 1.

0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 **4** 6 **4** 1
5	1 5 10 10 5 1
6	1 6 15 **20** 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 **28** **56** 17 **56** **28** 8 1
9	1 9 **36** **84** **126** **126** **84** **36** 9 1
10	1 10 **45** **120** 210 **252** 210 **120** **45** 10 1
11	1 11 55 165 330 462 462 330 165 55 11 1
12	1 **12** 66 **220** **495** **792** **924** **792** **495** **220** 66 **12** 1

What are the row numbers (printed in **bold**) of the rows which contain at least one bold entry: 4, 6, 8, 9, 10, 12, ...?

Example 40 We finally draw Pascal's triangle with $\binom{n}{r}$ in the nth row and $(n+r)$th column:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1																
1		1	1														
2			1	2	1												
3				1	3	3	1										
4					1	4	6	4	1								
5						1	5	10	10	5	1						
6							1	6	15	20	15	6	1				
7								1	7	21	35	35	21	7	1		
8									1	8	28	56	70	56	28	8	1
9										1	9	36	84	126	126	84	36
10											1	10	45	120	210	252	210
11												1	11	55	165	330	462
12													1	12	66	220	495

The column totals are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... These numbers also seem to appear in the next three examples, as well as in Examples 70 and 80.

Example 41 The **ceiling** of, least integer not less than, $e^{(n-1)/2}$, for

$n=$	0	1	2	3	4	5	6	7	8	9...
is	1	1	2	3	5	8	13	21	34	55....

Example 42 If $\sigma(n)$ is the sum of the divisors of n, then $\sigma(n)/n$ measures the **abundancy** of n. Every number n with abundancy $\geqslant j$ must have at least k factors, where, for

$j=$	2	3	4	5	6	7	8	9...
$k=$	2	3	4	6	9	14	22	35....

The differences of this last sequence are 1, 1, 2, 3, 5, 8, 13, ...

Example 43 You may have suspected that some of the sequences in the last three examples are manifestations of the ubiquitous Fibonacci numbers ($u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_{n+1} + u_n$). According to the Lucas-Lehmer theory [33] the **rank of apparition** (the least n for which p divides u_n) of a prime p in the Fibonacci sequence is a divisor of $p - (p|5)$, where $(p|5)$ is the Legendre symbol, 0 for $p = 5$, and $+1$ or -1 according as $p \equiv \pm 1$ or ± 2, mod 5, otherwise. For example, the rank of apparition for the first few primes is

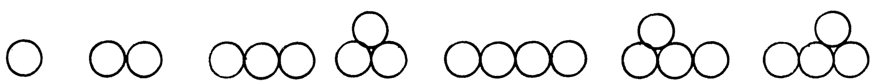
$p =$	2	3	5	7	11	13	17	19	23	29	31	37	41...
	3	4	5	8	10	7	9	18	24	14	30	19	20....

When a prime *does* first appear, does it always occur to the first power?

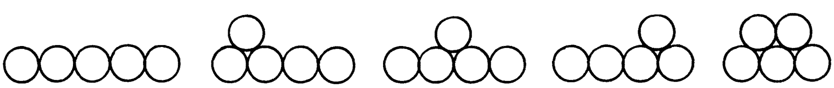
Example 44 Define a sequence by $c_1 = 1$, $c_2 = 2$ and c_{n+1} the least integer such that $c_{n+1} - c_{n-1}$ differs from all earlier positive differences $c_j - c_i$, $1 \leqslant i < j \leqslant n$, e.g.

$\{c_1, c_2\} = \{1,2\}$	difference 1	$c_3 - c_1 = 2$	$c_3 = 3$
$\{c_1, c_2, c_3\} = \{1,2,3\}$	differences 1, 2	$c_4 - c_2 = 3$	$c_4 = 5$
$\{c_1, \ldots, c_4\} = \{1,2,3,5\}$	differences 1, 2, 3, 4	$c_5 - c_3 = 5$	$c_5 = 8$
$\{c_1, \ldots, c_5\} = \{1,2,3,5,8\}$	differences $1, 2, \ldots, 7$	$c_6 - c_4 = 8$	$c_6 = 13$

Example 45 In the following arrangements of pennies, each row forms a contiguous block, and each penny above the bottom row touches two pennies in the row below it. Count such arrangements by the total number of pennies:



1 1 2 3



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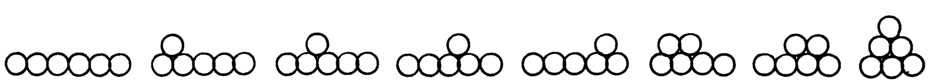

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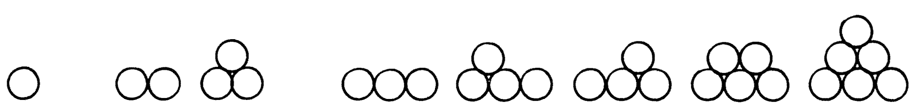
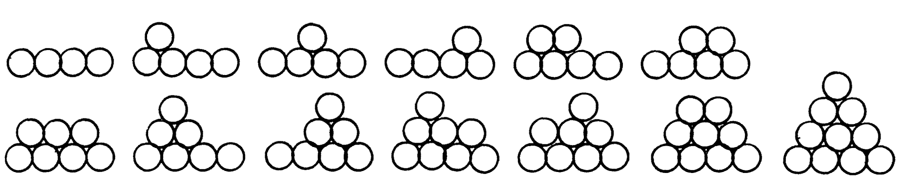
Example 46 Alternatively, you could count the arrangements in the previous example by the number of pennies in the bottom row.



1 2 5



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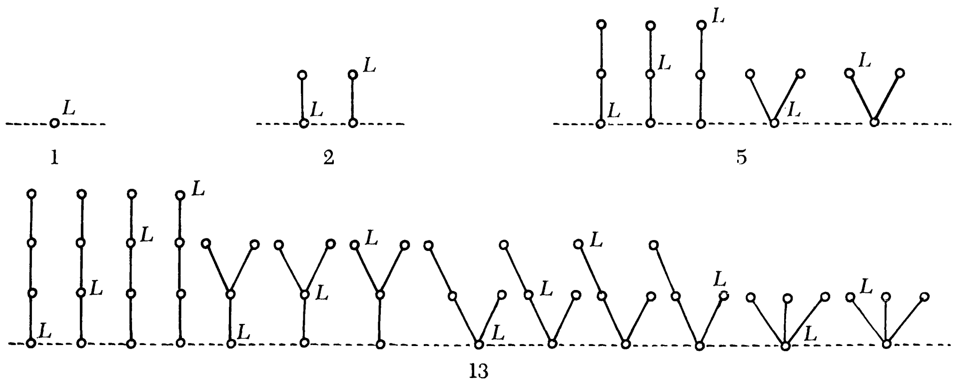
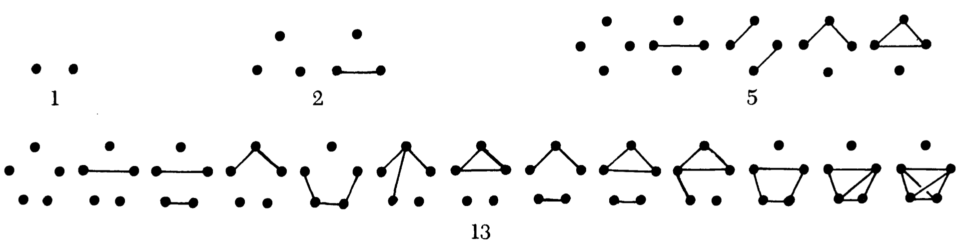
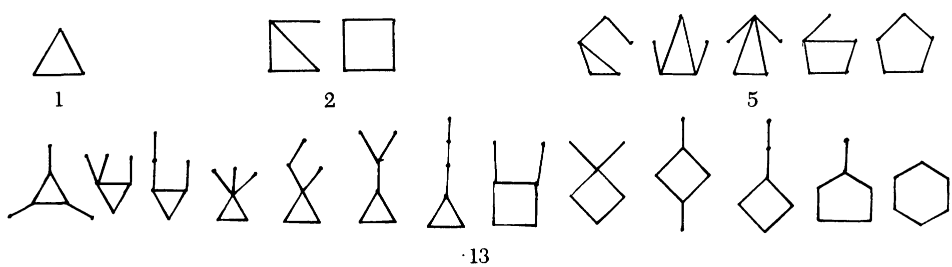
Example 47 The number of rooted trees with n vertices, just one of which is labelled.



Example 48 The number of disconnected graphs with $n + 1$ vertices.



Example 49 The number of connected graphs on $n + 2$ vertices with just one cycle.



For many other examples involving graphs, see [22], which does not, however, include Examples 47–49.

Example 50 The coefficients in the power series solution

$$y = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{2x^4}{4!} + \frac{5x^5}{5!} + \frac{13x^6}{6!} + \cdots$$

of the differential equation $D^2 y = e^x y$.

Example 51 The sequence $a_n = a_{n-1} + na_{n-2}$ $(n \geqslant 1)$ with $a_{-1} = a_0 = 1/2$

$$a_1 = \frac{1}{2} + 1 \times \frac{1}{2} = 1$$

$$a_2 = 1 + 2 \times \frac{1}{2} = 2$$

$$a_3 = 2 + 3 \times 1 = 5$$

$$a_4 = 5 + 4 \times 2 = 13.$$

Example 52 The sequence $b_n = (n-1)2^{n-2} + 1$, $n \geqslant 1$.

$$b_1 = 0 \times 2^{-1} + 1 = 1$$

$$b_2 = 1 \times 2^0 + 1 = 2$$

$$b_3 = 2 \times 2^1 + 1 = 5$$

$$b_4 = 3 \times 2^2 + 1 = 13.$$

Example 53 How many distinct sums, $f(n)$, may there be of n different ordinal numbers? Obviously, $f(1) = 1$. However, $f(2) = 2$, because ordinal addition is not commutative. For example, $1 + \omega = \omega \neq \omega + 1$. You might guess that $f(3)$ could be as large as $3! = 6$, but in fact you can't have more than 5 distinct sums of 3 different ordinals. The answers

for $n =$	1	2	3	4	5	6	7	8...
are $f(n) =$	1	2	5	13	33	81	193	449...

perhaps the same sequence as Example 52. Or perhaps not.

Example 54 The values of the polynomial $9n^2 - 231n + 1523$ for $n = 0, 1, 2, \ldots$ are 1523, 1301, 1097, 911, 743, 593, 461, 347, 251, 173, 113, 71, 47, 41, 53, 83, 131, 197,.... Try also the polynomial $47n^2 - 1701n + 10181$.

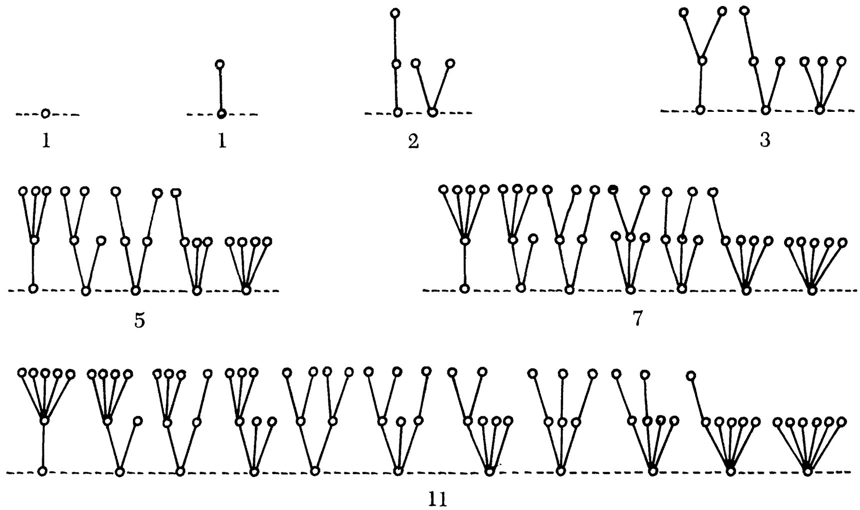
Example 55 What are the next three terms in the sequence

$$(1), 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, \ldots ?$$

Example 56 The integer part of the nth power of $3/2$

n	0	1	2	3	4	5	6
$(3/2)^n$	1	1.5	2.25	3.375	5.0625	7.59375	11.390625
	0	1	2	3	5	7	11

Example 57 The number of trees with n edges, and height at most 2.



Example 58 The number of partitions of n

$n=$	0	1	2	3	4	5	6	7	8	9...
$p(n)=$	1	1	2	3	5	7	11	15	22	30....

Example 59 If we form successive differences of the partition function:

1 1 2 3 5 7 11 15 22 30 42 56 77 101 135 176 231 297 385 490 627 ..
0 1 1 2 2 4 4 7 8 12 14 21 24 34 41 55 66 88 105 137 ...
1 0 1 0 2 0 3 1 4 3 7 3 10 7 14 11 22 17 32 ...
-1 1 -1 2 -2 3 -2 3 -2 5 -4 7 -3 7 -3 11 -5 15 ...

we see that the third-order differences alternate in sign.

Example 60 If you expand the product $(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$, you get, successively

$$1-x$$
$$1-x-x^2+x^3$$
$$1-x-x^2+x^4+x^5-x^6$$
$$1-x-x^2+2x^5-x^8-x^9+x^{10}$$

and a coefficient 2 has appeared. Indeed, at stage 10, a coefficient 3 appears. However, further calculation appears to cancel these out, leaving

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+\cdots.$$

Are there any coefficients other than 0, ± 1 in the final result?

Example 61 For each integer exponent, n, is there an integer $m > 1$ such that the sum of the decimal digits of m^n is equal to m?

$$2^1, 9^2 = 81, 8^3 = 512, 7^4 = 2401, 28^5 = 17210368, 18^6 = 34012224, 18^7 = 61222032, 46^8, 54^9, 82^{10}, 98^{11}, 108^{12}, 20^{13}, 91^{14}, 107^{15}, 133^{16}, 80^{17}, 172^{18}, 80^{19}, 90^{20}, 90^{21}, \ldots .$$

Example 62 A **Niven number** has been defined as one which is divisible by the sum of its decimal digits, such as 21 and 133. Is $n!$ always a Niven number?

$$4! = 24, 5! = 120, 6! = 720, 7! = 5040, 8! = 40320, 9! = 362880, 10! = 3628800, \ldots .$$

Example 63 Can you choose a sequence of real numbers from the interval $(0,1)$ so that the first two lie in different halves, the first three in different thirds, the first four in different quarters, and so on? For example,

$$0.71, 0.09, 0.42, 0.85, 0.27, 0.54, 0.925, 0.17, 0.62, 0.355, 0.78, 0.03, 0.48, \ldots .$$

If you run into difficulty, you are allowed to adjust earlier members of the sequence, if you like.

Example 64 Surely every odd number (greater than 1, if you don't want to count 1 as a prime) is expressible as a prime plus twice a square?

$$3 + 2 \cdot 0^2, 3 + 2 \cdot 1^2, 5 + 2 \cdot 1^2, 7 + 2 \cdot 1^2, 3 + 2 \cdot 2^2, 11 + 2 \cdot 1^2,$$
$$7 + 2 \cdot 2^2, 17 + 2 \cdot 0^2, 11 + 2 \cdot 2^2, 3 + 2 \cdot 3^2, 5 + 2 \cdot 3^2, 23 + 2 \cdot 1^2, \ldots .$$

Indeed, some numbers, such as 61, have several such representations.

Example 65 Is $n!$ always expressible as the difference of two powers of 2?

$$0! = 1! = 2^1 - 2^0, 2! = 2^2 - 2^1, 3! = 2^3 - 2^1, 4! = 2^5 - 2^3, 5! = 2^7 - 2^3, \ldots .$$

Example 66 It's well known that $4! = 5^2 - 1$, $5! = 11^2 - 1$ and $7! = 71^2 - 1$, but not so well known that if you take the *next* square bigger than $n!$ the difference is always a square:

$$6! = 27^2 - 3^2, 8! = 201^2 - 9^2, 9! = 603^2 - 27^2,$$
$$10! = 1905^2 - 15^2, 11! = 6318^2 - 18^2, \ldots .$$

Example 67 The values of $\sin^2(k\pi/12)$, for $k = 0, 1, \ldots 6$ are

$k =$	0	1	2	3	4	5	6
$\sin^2(k\pi/12) =$	0	$(2-\sqrt{3})/4$	$1/4$	$1/2$	$3/4$	$(2+\sqrt{3})/4$	1

It's also well known that

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{2n(2n-2)\cdots 4 \cdot 2} \frac{\pi}{2}.$$

If you calculate the integral by the trapezoidal rule, using 6 equal subintervals, you will get the answer

$$\left\{2^{2n-1} + (2+\sqrt{3})^n + (2-\sqrt{3})^n + 3^n + 2^n + 1\right\}\pi/12\cdot 4^n,$$

which is exact for $n = 1, 2, 3, 4, 5, 6$ and 7.

Example 68 The continued fraction for π^2/e^γ is

$$\frac{\pi^2}{e^\gamma} = 5 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots.$$

Example 69 Define a sequence by $P(1) = P(2) = 1$, and for $n > 2$, $P(n) = P(P(n-1)) + P(n - P(n-1))$. The first 32 terms are 1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7, 8, 8, 8, 8, 9, 10, 11, 12, 12, 13, 14, 14, 15, 15, 15, 16, 16, 16, 16, 16. Note that $P(2) = 1$, $P(4) = 2$, $P(8) = 4$, $P(16) = 8$, and $P(32) = 16$.

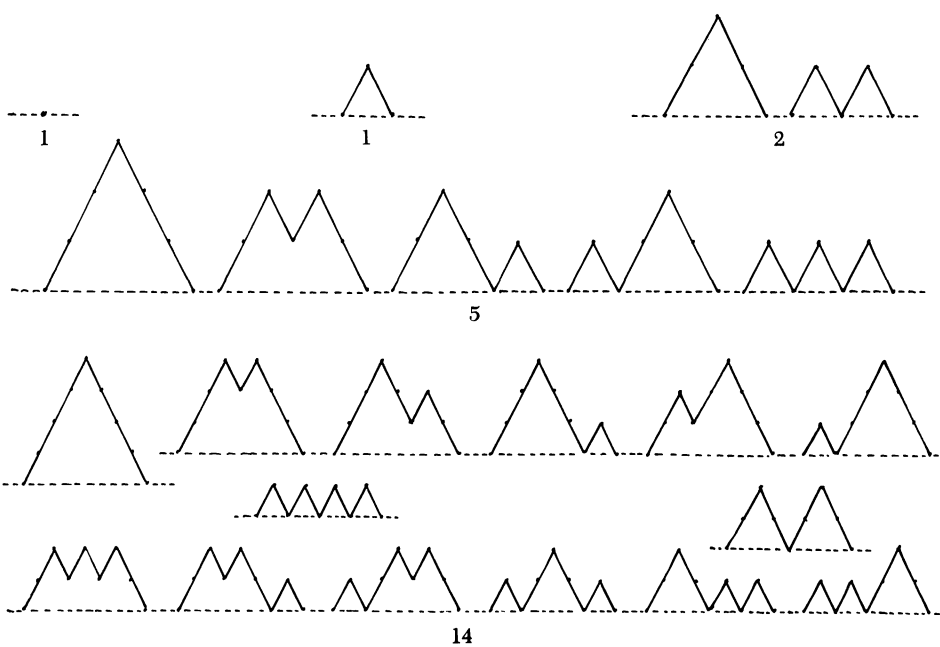
Example 70 A similar sequence starts with $Q(1) = Q(2) = Q(3) = 1$, and the same recurrence for $n > 3$, $Q(n) = Q(Q(n-1)) + Q(n - Q(n-1))$. The first 34 terms are 1, 1, 1, 2, 2, 3, 3, 3, 4, 5, 5, 5, 5, 6, 7, 7, 8, 8, 8, 8, 8, 9, 10, 11, 11, 12, 12, 12, 13, 13, 13, 13, 13, 13. Notice that $Q(2) = 1$, $Q(3) = 1$, $Q(5) = 2$, $Q(8) = 3$, $Q(13) = 5$, $Q(21) = 8$ and $Q(34) = 13$.

Examples 40–52 and 70 perhaps contain manifestations of the Fibonacci numbers. Almost as ubiquitous are the **Catalan numbers**, $(2n)!/n!(n+1)!$,

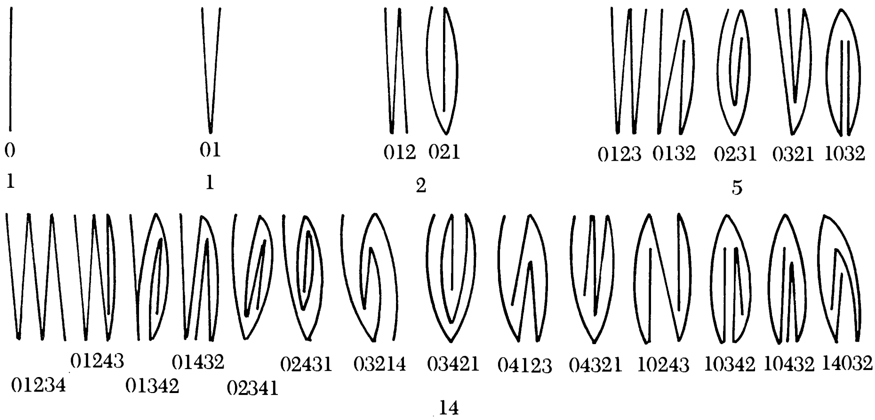
$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots.$$

How many of Examples 71 to 79 are genuine?

Example 71 The number of mountain ranges you can draw with n upstrokes and n downstrokes:



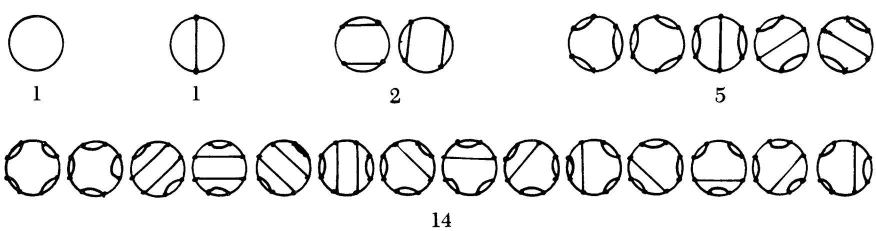
Example 72 The number of ways of making n folds in a strip of $n+1$ postage stamps, where we don't distinguish between front and back, top and bottom, or left and right:



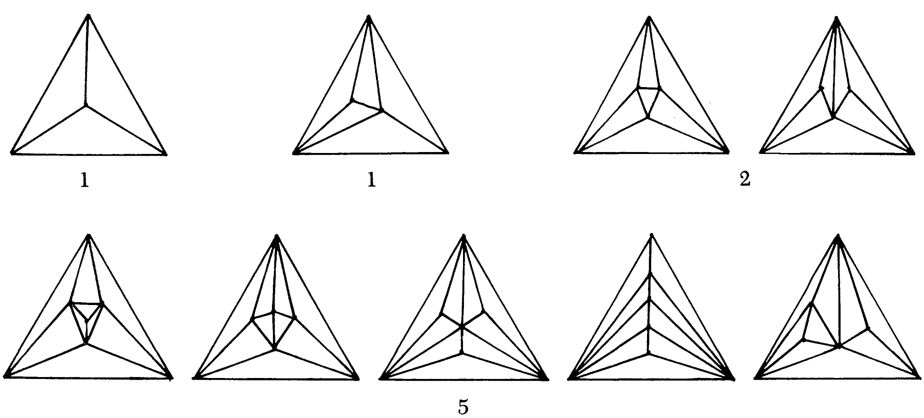
Example 73 The number of different groups, up to isomorphism, of order 2^n is,

for $n =$	0	1	2	3	4...
no. of groups $=$	1	1	2	5	14....

Example 74 The number of ways $2n$ people at a round table can shake hands in pairs without their hands crossing.

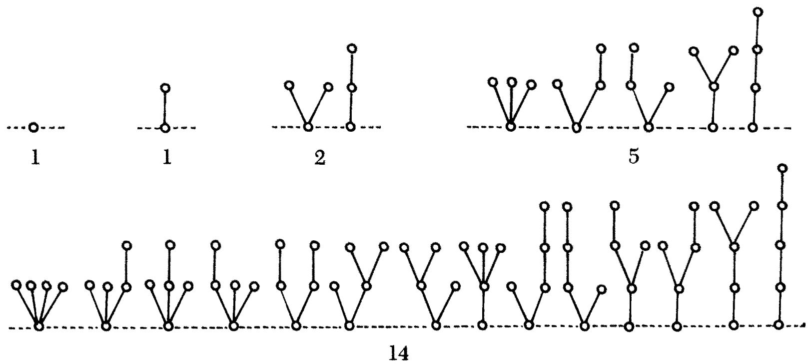


Example 75 The number of triangulations of the sphere with $n+4$ points.

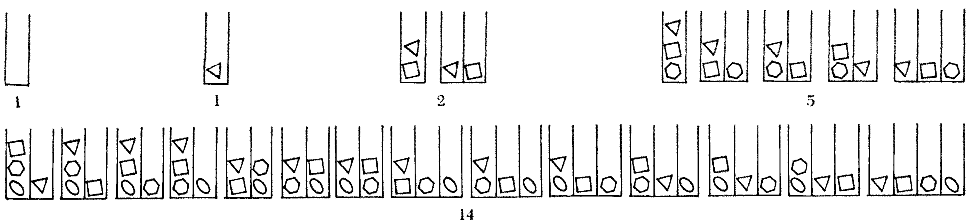


We leave the reader to verify that there are just 14 distinct triangulations of the sphere with 8 points.

Example 76 The number of rooted plane trees with n edges. "Plane" means that we distinguish between left and right.



Example 77 The number of ways of distributing n different objects in indistinguishable boxes, with at most three objects in a box.



Example 78 The probability, for $n = 0, 1, 2, \ldots$, that a function $f(k)$, with domain $k = 0, 1, \ldots, n$ and range $[0, 1]$, is convex, is

$$\frac{1}{(0!)^2}, \frac{1}{(1!)^2}, \frac{2}{(2!)^2}, \frac{5}{(3!)^2}, \frac{14}{(4!)^2}, \ldots .$$

Example 79 The incomplete Bessel function of order one has power series expansion

$$I_1(2x)/x = 1 + \frac{x^2}{2!} + \frac{2x^2}{4!} + \frac{5x^6}{6!} + \frac{14x^8}{8!} + \cdots .$$

Example 80 Examples 37 to 40 involved Pascal's triangle, whose entries are the binomial coefficients. We can use a similar array to expand $(1 + x + x^2)^n$, giving **trinomial coefficients**. Each entry is the sum of the three nearest in the previous row.

									1									
								1	1	1								
							1	2	3	2	1							
						1	3	6	7	6	3	1						
					1	4	10	16	19	16	10	4	1					
				1	5	15	30	45	51	45	30	15	5	1				
			1	6	21	50	90	126	141	126	90	50	21	6	1			
		1	7	28	77	161	266	357	393	357	266	161	77	28	7	1		
	1	8	36	112	266	504	784	1016	1107	1016	784	504	266	112	36	8	1	
1	9	45	156	414	882	1554	2304	2907	3139	2907	2304	1554	882	414	156	45	9	1

The central trinomial coefficient, a_n,

$$1,1,3,7,19,51,141,393,1107,3139,\ldots$$

almost trebles in size at each step: if we calculate $3a_n - a_{n+1}$ we get

$$2,0,2,2,6,12,30,72,182,\ldots$$

which are **pronic numbers**, $m(m+1)$, for $m = 1,0,1,1,2,3,5,8,13,\ldots$.

Answers

36. This is the polynomial $\binom{n-1}{0}+\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\binom{n-1}{4}$ of Example 5 of [**21**], and represents the number of pieces you can cut a circular cake into by slicing between every pair of points chosen from n around the circumference. It is also the number of regions that 4-dimensional space is chopped into by $n-1$ hyperplanes in general position. The sequence is #427 in [**45**]: 1, 2, 4, 8, 16, 31, 57, 99, 163, 256, 386, 562, 794, 1093, 1471, ...
37. This very beautiful setting for Example 1 of [**21**] was observed 20 years ago by William Watkins, now co-editor of *Coll. Math. J.* Gauss has told us that the number of sides in a regular polygon which can be constructed with straightedge and compass is of shape $2^m \Pi F_n$, where the F_n are distinct Fermat primes $2^{2^n}+1$. Only five such, $0 \leqslant n \leqslant 4$, are known and some people believe that no others will ever be found. So the pattern breaks down at row 32. Fermat thought that $2^{32}+1$ was prime, but Euler discovered the factorization 641×6700417.
38. This is the Mann-Shanks primality test [**36**]. Surprising, if not practical. Can you prove it?
39. This is an observation of Gerry Myerson: that the bold numbers are the composite numbers. However, this breaks down in row 13, because $\binom{13}{5} = 3^2 11 \cdot 13$ and $\binom{13}{6} = 2^2 3 \cdot 11 \cdot 13$ are not squarefree.
40. This well-known relation between Pascal's triangle and Fibonacci numbers is easily seen to persist, since each entry is the sum of the entries in the previous two columns of the previous row, so each total is the sum of the two previous totals.
41. This is adapted from an inequality of Larry Hoehn, of Clarksville TN. The coincidence is quite surprising, since $\sqrt{e} \approx 1.64872$ and the golden ratio $(1+\sqrt{5})/2 \approx 1.61803$ are not remarkably close. For $n = 10,11,12,\ldots$ the terms $91,149,245,\ldots$ begin to diverge from the Fibonacci sequence $89,144,233,\ldots$.
42. In [**32**], Richard Laatsch shows that the sequence continues $55,89,142,230,\ldots$ with differences

    20 30 53 88 143 236 387 641 1061 1763 2737 4903 8202 13750 23095...

    which stay close to the Fibonacci numbers

    21 34 55 89 144 233 377 610 987 1597 2584 4181 6765 10946 17711...

    for awhile, but eventually tend to infinity more rapidly.
43. See sequence #912 in [**45**]. This is still a notorious open question: there are extensive tables [**30, 34, 49, 50**]. During revision of this article, Dick Lehmer kindly ran a program on a 75 Vax, and found no counterexample with p less than a million.
44. The sequence continues $17,26,34,45,54,67,\ldots$ and is denser than the Fibonacci sequence. It is #254 in [**45**], but the reference there is misleading. The sequence doesn't solve *Amer. Math. Monthly* problem E1910 [1966, 775; partial solution

1968, 80–81] because the differences are not unique: e.g., $17-8=26-17=54-45$. Nor is it the auxiliary sequence $\{r_n\}=\{4, 5, 9, 10, 11, 16, 18, 22, 23, 24, 25, 27, 28, 29,\dots\}$, used to construct the Sierpiński sequence, #425 in [**45**]. There's another open question here: find the smallest possible asymptotic growth for a sequence of integers such that every positive integer occurs *uniquely* as a difference.

45. This also fails to continue with the Fibonacci sequence. The numbers of arrangements with $7,8,9,\dots$ pennies are $12,18,26,\dots$. These arrangements were studied by Auluck [**2**]; see sequence #253 in [**45**], and compare Example 34 in [**21**].
46. These are indeed the odd-ranking Fibonacci numbers, u_{2n-1}, sequence #569 in [**45**], which have the property

$$u_{2n-1}=u_{2n-3}+2u_{2n-5}+3u_{2n-7}+\cdots+(n-1)u_1+1$$

which can be seen to be the number of ways that a row of n pennies may be surmounted by an arrangement with $n-k$ in its bottom row, in any one of k possible positions, where $k=1,2,\dots,n-1$ or it's not surmounted at all $(k=n)$.
47. These are *not* the alternate Fibonacci numbers, e.g., the numbers of such trees with $5,6,7,\dots$ vertices are $35,95,262,\dots$ See sequence #570 in[**45**] or p. 134 in [**43**].
48. Nor are these. The next few members of the sequence are 44, 191, 1229, 13588, 288597,.... See sequence #574 in [**45**], or [**24**].
49. Neither is this the sequence of alternate Fibonacci numbers, but continues 33 (one short!), 89 (correct!), 240 (7 too many), 657, 1806, 5026,.... See sequence #568 in [**45**] or page 150 in [**43**].
50. Nor is this, which continues $36,109,359,1266,4731,18657,77464,\dots$; see sequence #572 in [**45**] and Tauber's paper [**48**].
51. Nor again, since $a_5=13+5\times5=38$, $a_6=116$, $a_7=382,\dots$; see sequence #573 in [**45**].
52. Neither are these, $b_5=4\cdot2^3+1=33$, $b_6=5\cdot2^4+1=81$, $b_7=6\cdot2^5+1=193$, $b_8=7\cdot2^6+1=449$, alternate Fibonacci numbers, but they *do* feature (for a while) in the next Example:
53. which I got from John Conway. If $g(k)=k\cdot2^{k-1}+1$, then

$$f(n)=\max_{0<k<n} f(n-k)g(k),$$

and, for $n\leqslant 8$, $f(n)$ is indeed equal to $g(n-1)$. Thereafter the situation gets more complicated, but a simple rule eventually emerges: for $n=9,10,11,12,13$, $f(n)=33^2,33\cdot81,81^2,81\cdot193,193^2$, and, for $n\geqslant14$, $f(n)=81f(n-5)$, except that $f(19)=193^3$.
54. Several readers of [**21**] said that I should have included Euler's famous formula, n^2+n+41, which gives primes for $0\leqslant n\leqslant39$, not noticing that Example 21 was just that, except for the disguise of omitting the tell-tale 41 $(n=0)$. For some astonishing examples of The Strong Law in this connection, see the papers of Stark [**46, 47**]. The present polynomial is a slight adaptation of one due to Sidney Kravitz, and is found by replacing n in Euler's formula by $38-3n$. Surprisingly, this still gives primes for $0\leqslant n\leqslant39$, although thirteen of them are not among the original forty; $n=40$ and 41 give $6683=41\times163$ and $7181=43\times167$.

The polynomial $47n^2-1701n+10181$ was discovered recently by Gilbert Fung. If you work modulo p for primes $2\leqslant p\leqslant43$, you'll find that it's never divisible by such primes. It takes prime values for $0\leqslant n\leqslant42$, beating Euler's record by two. Notice that the discriminant of Euler's polynomial is -163, and that of Kravitz is $-3^2 \times 163$, while Fung's polynomial has to have a *positive* discriminant, 979373.

55. Such questions are hardly fair, since arguments can be advanced for continuing sequences in any way you wish. Some answers are more plausible than others, however, and the one that Persi Diaconis hoped you would miss is $59, 60, 61, \ldots$, the orders of the simple groups!
56. Another futile attempt to fool you into thinking of the primes. The next member is 17, then $25, 38, \ldots$; see sequence #245 in [**45**].
57. This is not the same sequence as the previous example, but see the next!
58. To see the correspondence between this and the previous example, note that the number of vertices at height one is the number of parts, and the valences of these vertices are the sizes of the parts. Sequence #244 in [**45**]; see also page 122 in [**43**] and page 836 in [**1**].
59. This example was sent by Gerry Myerson. It can be proved that the differences of any order are positive from some point on, but that point recedes rather rapidly as you take higher order differences. The next few third differences are $-4, 17, -2, 24, -4, 32, 1, 38, 5, \ldots$ and are positive from now on. The fourth differences alternate in sign until the 67th, after which they are positive.
60. This is Euler's famous pentagonal numbers theorem:

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2}.$$

See theorem 353 in [**26**], for example.

61. Norman Megill of Waltham, MA, finds such m for each $n \leqslant 104$. For $n = 105$, however, no such m exists.
62. This question was asked by Sam Yates. Carl Pomerance suggested that counterexamples might be expected by the time n has reached 500, and indeed Yates found that 432! is not a Niven number, since the sum of its digits is $3^2 \times 433$, and 433 is prime.
63. The given sequence can be continued, $0.97, 0.22, 0.66, 0.32$, but Berlekamp and Graham [3] have shown that no such sequence exists with more than 17 members!
64. This special case of the Hardy-Littlewood problem was mentioned by Ron Ruemmler of Edison, NJ, who believes that the first exception is 5777, and asks if it is also the last! It is known from the work of Hooley [**27**], Miech [**37**], and Polyakov [**42**] that the density of exceptions is zero.
65. Ignace Kolodner got this from Harold N. Shapiro in an NYU Problem Seminar in 1949. It's left to the reader to prove that $n!$ is never again the difference of two powers of two.
66. This was observed by Larry Hoehn of Clarksville, TN. It fails for 12!, but $13! = 78912^2 - 288^2$, $14! = 295260^2 - 420^2$, $15! = 1143536^2 - 464^2$, $16! = 4574144^2 - 1856^2$. It's doubtful if this often occurs from here on (note that you must take the *next* square bigger than $n!$), but it may be hard to prove anything.
67. This is also correct for $n = 8$, 9, 10, and 11, but for $n = 12$ we get $(1352079\pi)/2^{24}$ instead of $(1352078\pi)/2^{24}$, out by 3 parts in four million! The trapezoidal rule gives the right answer if you use k subintervals, provided $2n$ is less than $4k$: see [**28**], for example. David Bloom suggested that "four million" should read "sixteen million": I intended the *relative* error, $\approx 2.958/4000000$: the *actual* error is $\approx 2.996/16000000$: more examples of the Strong Law!
68. If this pattern, noticed by James Conlan [8], were to continue, we would have $(5 + \sqrt{37})e^{\gamma} = 2\pi^2$. Close, but no cigar!

69. The sequence that hit the national presses on both sides of the Atlantic, e.g. [**6**], publicizing the Conway-Mallows encounter. I have an earlier manuscript of Conway in which he has written (in another notation) "$P(2^k) = 2^{k-1}$ (easy), $P(2n) \leqslant 2P(n)$ (hard), $P(n)/n \to \frac{1}{2}$ (harder)." It was the proof of a precise form of this last statement that almost won Mallows even more money than Conway intended. Papers mentioning this sequence include [**16**, **35**].
70. Yes, the Fibonacci pattern continues [**40**]. David Newman showed this to David Bloom as a conjecture in 1986.

Nine of the final ten examples are intended to look like the Catalan numbers; sequence #577 in [**45**]. At first it is a matter of some surprise that

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

is always an integer. In connection with some recent correspondence [**41**], John Conway makes the more general observation that

$$\frac{(m,n)(m+n-1)!}{m!n!}$$

is an integer, where (m, n) is the g.c.d. of m and n, because

$$\frac{m(m+n-1)!}{m!n!} = \binom{m+n-1}{m-1} \quad \text{and} \quad \frac{n(m+n-1)!}{m!n!} = \binom{m+n-1}{n-1}$$

are both integers. This also answers a question in B33 of [**20**], where Neil Sloane gave the example $n = 4m + 3$.

Catalan numbers occur in many widely different looking contexts: see [**18**], with nearly 500 references, and [**31**], with a list of 31 structures, both obtainable from H. W. Gould, Department of Mathematics, West Virginia University, Morgantown, WV, 26506. An article with a good bibliography is [**5**]. Several "proofs without words," showing the equivalence of several of the structures, will appear in [**9**].

71. This is a genuine example of the Catalan numbers. The mountain ranges are the same as paths from $(0,0)$ to (n, n) which do not cross $y = x$, or incoming tied ballots in which one candidate is never behind, or sequences of zeros and ones, or of ± 1s, subject to appropriate sum conditions, e.g., random one-dimensional walks in which you never go to the left of the origin; see [**13**].
72. This sequence, #576 in [**45**], is not, and continues 39 (not 38, as stated in [**14**]), 120, 358, 1176, 3527, 11622, 36627, 121622, 389560, ..., see [**29**].
73. The numbers of groups of orders 2^5 and 2^6 are 51 and 267 [**23**]. This sequence, #581 in [**45**], continues 2328 [**51**], 56092 [**52**].
74. This is genuine Catalan again: see [**39**].
75. But this one, sequence #580 in [**45**], has been calculated for only four more terms [**4**, **12**, **19**]. Of the

$$1, 1, 2, 5, 14, 50, 233, 1249, 7595 \text{ triangulations},$$

only $0, 0, 1, 1, 2, 5, 12, 34, 130$ contain no vertex of valence 3.
76. is a genuine manifestation of the Catalan numbers [**7**, **25**], but
77. is not: sequence #579 in [**45**] continues 46, 166, 652, 2780, 12644, 61136, 312676, 1680592, ... [**38**].
78. The probability for general n is indeed $c_n/(n!)^2$ [**10**].
79. In [**10**] we asked what was the exponential generating function for the Catalan

numbers. Louis W. Shapiro observes that

$$\sum_{n=0}^{\infty} c_n \frac{x^{2n}}{(2n)!} = I_1(2x)/x$$

where I_1 is the modified Bessel function of order one: see formula 9.6.10. on page 375 of [**1**]. In the paper [**44**] he obtains results for lattice paths which stay below given points, arranged with increasing abscissas and ordinates, somewhat analogous to the convex functions of [**10**].

Before we say goodbye to the Catalan numbers, here's an observation which may not be widely known. It originated in a discussion with John Conway only six months ago. What *is* well known is that the Catalan numbers are associated with parenthesization. By that most people mean the numbers of possible orders of n nonassociative operations, usually indicated by $n-1$ pairs of parentheses:

$n=0$	a	$n=1$ ab	$n=2$	$(ab)c$ or	$a(bc)$
$n=3$	$((ab)c)d$	$(a(bc))d$	$a((bc)d)$	$a(b(cd))$	$(ab)(cd)$
$n=4$	$(((ab)c)d)e$	$((a(bc))d)e$	$(a((bc)d))e$	$(a(b(cd)))e$	$((ab)(cd))e$
	$((ab)c)(de)$	$(a(bc))(de)$	$(ab)((cd)e)$	$(ab)(c(de))$	$a(((bc)d)e)$
	$a((b(cd))e)$	$a((bc)(de))$	$a(b((cd)e))$	$a(b(c(de)))$	

and so on. But they are also the numbers of ways of arranging n pairs of parentheses as a pattern, just for their own sake:

$n=0$ $n=1$ ( ) $n=2$ (( )) or ( )( )

$n=3$ ((( ))) (( ))( ) ( )(( )) (( )( )) ( )( )( )

$n=4$ (((( )))) ((( )))( ) ( )((( ))) ((( )( ))) ((( ))( )) (( )(( ))) (( ))(( ))

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An examination of the symmetries in the two cases makes it unlikely that you'll find a direct combinatorial comparison. One-one correspondences between the former manifestation and other Catalan manifestations are well known. The latter are easily seen to be in correspondence with the pairs of people shaking hands in Example 73, and with the mountains in Example 70.

80. Jack Good [**17**] has given an asymptotic formula for the central trinomial coefficient:

$$a_n \sim \frac{3^{n+\frac{1}{2}}}{2\sqrt{\pi n}}\left\{1 - \frac{3}{16n} + \frac{1}{512n^2} + O(n^{-3})\right\}$$

which shows that the left side of the "identity"

$$¿ \quad 3a_n - a_{n+1} = u_{n-1}(u_{n-1}+1) \quad ?$$

grows like $c \times 3^n \times n^{-3/2}$, whereas the right side grows like $\tau^{2n}/5$, where τ is the golden ratio, $\tau^2 = (3+\sqrt{5})/2$. Further calculation shows that $a_{10} = 8953$, $3a_9 - a_{10} = 464$, while $u_8(u_8+1) = 21 \times 22 = 462$. The asymptotic formula is good to the nearest integer for quite large values of n.

This example was sent by Donald Knuth. Euler [**11**] was one of the earlier discoverers of The Strong Law of Small Numbers, and called this

exemplum memorabile inductionis fallacis.

On the same page he gives the Fibonacci formula that's often attributed to Binet.

Coda I showed this example to George Andrews during the recent Bateman Retirement Conference at Allerton Park, Illinois. Half-an-hour later he came back with what Euler really should have said. He defines the trinomial coefficients centrally by

$$(1+x+x^2)^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^{n+j}$$

and proves that, if F_n is the nth Fibonacci number, then

$$F_n(F_n+1) = 2\sum_{\lambda=-\infty}^{\infty}\left(\binom{n}{10\lambda+1}_2 - \binom{n}{10\lambda+2}_2\right).$$

For $-1 \leqslant n \leqslant 7$, the only nonzero term on the right is $\lambda = 0$, which accounts for Euler's observation, since

$$3\binom{n}{0}_2 - \binom{n+1}{0}_2 = 2\binom{n}{0}_2 - 2\binom{n}{1}_2.$$

Andrews will publish the q-analog of this theorem shortly.

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NOTES

A Fibonacci-like Sequence of Composite Numbers

DONALD E. KNUTH*
Stanford University
Stanford, CA 94305

Ronald L. Graham [1] found relatively prime integers a and b such that the sequence $\langle A_0, A_1, A_2, \dots \rangle$ defined by

$$A_0 = a, \qquad A_1 = b, \qquad A_n = A_{n-1} + A_{n-2} \tag{1}$$

contains no prime numbers. His original method proved that the integers

$$a = 331635635998274737472200656430763$$
$$b = 1510028911088401971189590305498785 \tag{2}$$

have this property. The purpose of the present note is to show that the smaller pair of integers

$$a = 62638280004239857$$
$$b = 49463435743205655 \tag{3}$$

also defines such a sequence.

Let $\langle F_0, F_1, F_2, \dots \rangle$ be the Fibonacci sequence, defined by (1) with $a = 0$ and $b = 1$; and let $F_{-1} = 1$. Then

$$A_n = F_{n-1}a + F_n b. \tag{4}$$

Graham's idea was to find eighteen triples of numbers (p_k, m_k, r_k) with the properties that

- p_k is prime;
- F_n is divisible by p_k iff n is divisible by m_k;
- every integer n is congruent to r_k modulo m_k for some k.

He chose a and b so that

$$a \equiv F_{m_k - r_k}, \qquad b \equiv F_{m_k - r_k + 1} \pmod{p_k}. \tag{5}$$

It followed that

$$A_n \equiv 0 \pmod{p_k} \quad \Leftrightarrow \quad n \equiv r_k \pmod{m_k} \tag{6}$$

for all n and k. Each A_n was consequently divisible by some p_k; it could not be prime.

The eighteen triples in Graham's construction were

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$$\begin{array}{lll} (3,4,1) & (2,3,2) & (5,5,1) \\ (7,8,3) & (17,9,4) & (11,10,2) \\ (47,16,7) & (19,18,10) & (61,15,3) \\ (2207,32,15) & (53,27,16) & (31,30,24) \\ (1087,64,31) & (109,27,7) & (41,20,10) \\ (4481,64,63) & (5779,54,52) & (2521,60,60) \end{array} \qquad (7)$$

(It is easy to check that the second property above holds, because m_k is the first subscript such that F_{m_k} is divisible by p_k. The third property holds because the first column nicely "covers" all odd values of n; the middle column covers all even n that are not divisible by 6; the third column covers all multiples of 6.) It is not difficult to verify by computer that the values of a and b in (2) satisfy (5) for all eighteen triples (7); therefore, by the Chinese remainder theorem, these values are the smallest nonnegative integers that satisfy (5) for $1 \leqslant k \leqslant 18$. Moreover, these huge numbers are relatively prime, so they produce a sequence of the required type.

Incidentally, the values of a and b in (2) are not the same as the 34-digit values in Graham's original paper [1]. A minor slip caused his original numbers to be respectively congruent to F_{32} and F_{33} (mod 1087), not to F_{33} and F_{34}, although all the other conditions were satisfied. Therefore the sequences defined by his published starting values may contain a prime number A_{64n+31}. We are fortunate that calculations with large integers are now much simpler than they were in the early 60s when Graham originally investigated this problem.

But we need not use the full strength of (5) to deduce (6). For example, if we want

$$A_n \equiv 0 \pmod 3 \quad \Leftrightarrow \quad n \equiv 1 \pmod 4,$$

it is necessary and sufficient to choose $a \not\equiv 0 \pmod 3$ and $b \equiv 0 \pmod 3$; we need not stipulate that $a \equiv 2$ as required by (5). Similarly if we want

$$A_n \equiv 0 \pmod{17} \quad \Leftrightarrow \quad n \equiv 4 \pmod 9$$

it is necessary and sufficient to have

$$A_4 \equiv 0 \pmod{17} \quad \text{and} \quad A_5 \not\equiv 0 \pmod{17};$$

the sequence $\langle A_4, A_5, A_6, \ldots \rangle$ will then be, modulo 17, a nonzero multiple of the Fibonacci sequence $\langle F_0, F_1, F_2, \ldots \rangle$. The latter condition can also be rewritten in terms of a and b,

$$b \equiv 5a \pmod{17} \quad \text{and} \quad a \not\equiv 0 \pmod{17},$$

because $A_4 = 2a + 3b$ and $A_5 = 3a + 5b$. This pair of congruences has 16 times as many solutions as the corresponding relations $a \equiv 5$ and $b \equiv 8$ in (5).

Proceeding in this way, we can recast the congruence conditions (6) in an equivalent form

$$b \equiv d_k a \pmod{p_k} \quad \text{and} \quad a \not\equiv 0 \pmod{p_k}, \qquad (8)$$

for each of the first seventeen values of k. We choose d_k so that

$$F_{r_k - 1} + d_k F_{r_k} \equiv 0 \pmod{p_k};$$

this can be done since $0 < r_k < m_k$, hence F_{r_k} is not a multiple of p_k. The following pairs (p_k, d_k) are obtained:

(3,0)	(2,1)	(5,0)
(7,3)	(17,5)	(11,10)
(47,3)	(19,17)	(61,30)
(2207,3)	(53,4)	(31,21)
(1087,3)	(109,100)	(41,21)
(4481,1)	(5779,2)	(2521, *)

In each case we have

$$F_{r_k} + d_k F_{r_k+1} \not\equiv 0 \pmod{p_k}.$$

(Otherwise it would follow that $F_n + d_k F_{n+1} \equiv 0$ for all n and we would have a contradiction when $n = 0$.)

The final case is different, because $r_{18} = m_{18}$. We want

$$a \equiv 0 \pmod{2521} \quad \text{and} \quad b \not\equiv 0 \pmod{2521} \tag{9}$$

in order to ensure that the numbers A_{60n} are divisible by 2521.

Let us therefore try to find "small" integers a and b that satisfy (8) and (9). The first step is to find an integer D such that

$$D \equiv d_k \pmod{p_k} \tag{10}$$

for $1 \leqslant k \leqslant 17$. Then (8) is equivalent to

$$b \equiv Da \pmod{P} \quad \text{and} \quad \gcd(a, P) = 1, \tag{11}$$

where

$$P = p_1 p_2 \cdots p_{17} = 9757748694274371001434366645870. \tag{12}$$

Such an integer D can be found by using the Chinese remainder algorithm (see, for example, Knuth [2, Section 4.3.2]); it is

$$D = -2548019807824558291186694889 75, \tag{13}$$

uniquely determined modulo P.

Our goal is now to find reasonably small positive integers a and b such that

$$a = 2521n, \qquad b = aD \bmod P, \tag{14}$$

for some integer n. If a and b are also relatively prime, we will be done, because (8) and (9) will hold.

Let $C = 2521D \bmod P$. We can solve (14) in principle by trying the successive values $n = 1, 2, 3, \ldots$, looking for small remainders $b = nC \bmod P$ that occur before the value of $a = 2521n$ gets too large. In practice, we can go faster by using the fact that the smallest values of $nC \bmod P$ can be computed from the continued fraction for C/P (or equivalently from the quotients that arise when Euclid's algorithm is used to find the greatest common divisor of C and P).

Namely, suppose that Euclid's algorithm produces the quotients and remainders

$$\begin{aligned} P_0 &= q_1 P_1 + P_2, \\ P_1 &= q_2 P_2 + P_3, \\ P_2 &= q_3 P_3 + P_4, \ \ldots \end{aligned} \tag{15}$$

when $P_0 = P$ and $P_1 = C$. Let us construct the sequence

$$n_0 = 1, \qquad n_1 = q_1, \qquad n_j = q_j n_{j-1} + n_{j-2}. \tag{16}$$

Then it is well known (and not difficult to prove from scratch, see Knuth [3, exercise 6.4–8]) that the "record-breaking" smallest values of $nC \bmod P$ as n increases, starting at $n = 1$, are the following:

n	$nC \bmod P$	
$kn_1 + n_0$	$P_1 - kP_2$	for $0 \leqslant k \leqslant q_2$
$kn_3 + n_2$	$P_3 - kP_4$	for $0 \leqslant k \leqslant q_4$
$kn_5 + n_4$	$P_5 - kP_6$	for $0 \leqslant k \leqslant q_6$

and so on. (Notice that when, say, $k = q_4$, we have $kn_3 + n_2 = n_4$ and $P_3 - kP_4 = P_5$; so the second row of this table overlaps with the case $k = 0$ of the third row. The same overlap occurs between every pair of adjacent rows.) In our case we have

$$\langle q_1, q_2, q_3, \ldots \rangle = \langle 1,2,3,2,1,3,28,1,4,1,1,1,6,12626,1,195,4,7,1,1,2,\ldots \rangle \tag{17}$$

and it follows that

$$\langle n_1, n_2, n_3, \ldots \rangle = \langle 1,3,10,23,33,122,\ldots \rangle.$$

The record-breaking values of $nC \bmod P$ begin with

n	$nC \bmod P$
1	679845400109903786358967922355
2	383915930792370472574499198840
3	87986461474837158790030475325
13	56016376581815321375653177785
23	24046291688793483961275880245

These special values of n increase exponentially as the values of $nC \bmod P$ decrease exponentially.

The "best" choice of $a = 2521n$ and $b = nC \bmod P$, if we try to minimize $\max(a, b)$, is obtained when a and b are approximately equal. This crossing point occurs among the values $n = kn_{17} + n_{16}$, for $0 \leqslant k \leqslant q_{18} = 7$, when we have

$a = 2521n$	$b = nC \bmod P$	$\gcd(a, b)$
2502466953682069	237607917830996295	11
12525102462108367	206250504149697855	1
22547737970534665	174893090468399415	35
32570373478960963	143535676787100975	1
42593008987387261	112178263105802535	17
52615644495813559	80820849424504095	1
62638280004239857	49463435743205655	1
72660915512666155	18106022061907215	5

We must throw out cases with $\gcd(a, b) \neq 1$, but (luckily) this condition doesn't affect the two values that come nearest each other. The winning numbers are the 17-digit values quoted above in (3).

Slight changes in (7) will probably lead to starting pairs (a, b) that are slightly smaller than the 17-digit numbers in (3). But a proof applicable to substantially smaller starting values, with say fewer than ten digits each, would be quite remarkable.

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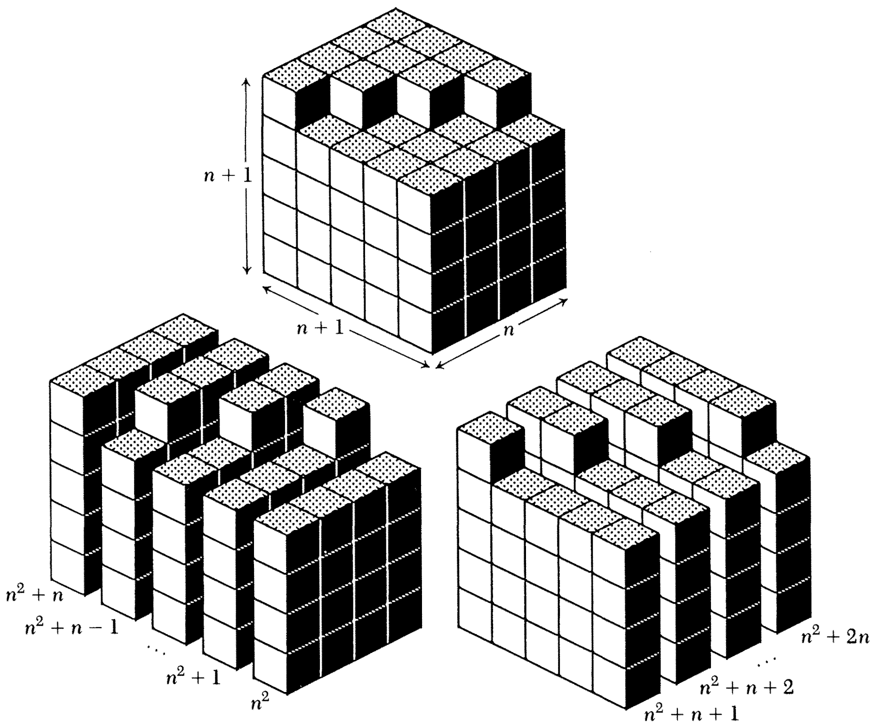
Proof without Words: Consecutive sums of consecutive integers



$$1+2=3$$
$$4+5+6=7+8$$
$$9+10+11+12=13+14+15$$
$$16+17+18+19+20=21+22+23+24$$
$$\vdots$$
$$n^2+(n^2+1)+\cdots+(n^2+n)=(n^2+n+1)+\cdots+(n^2+2n)$$

—Roger B. Nelsen
Lewis and Clark College
Portland, OR 97219

A River-Crossing Problem in Cross-Cultural Perspective

MARCIA ASCHER
Ithaca College
Ithaca, NY 14850

1. Introduction Most mathematicians react with interest to the challenge of a logical puzzle. In fact, some story puzzles have become such favorites that many of us cannot even recall where we learned them. Perhaps one of the best known is the puzzle in which a man must ferry across a river a wolf, a goat, and a head of cabbage. The difficulty is that the available boat can only carry him and one other thing but *neither the wolf and goat nor goat and cabbage can be left alone together.* Story puzzles are simple and accessible because they do not rely on any particular body of knowledge and yet they are mathematical in that a stated goal must be achieved under a given set of logical constraints. Attention to logic, as evidenced by the existence of these puzzles, is not the exclusive province of any one culture or subculture. Here, the river-crossing problem, in African cultures as well as in Western culture, will be used as an explicit example of the panhuman concern for mathematical ideas. Story puzzles are expressions of their cultures and so variations will be seen in the characters, the settings and the way in which the logical problem is framed.

2. Western versions The Western origin of the wolf, goat, and cabbage puzzle is most often attributed to a set of 53 problems designed to challenge youthful minds, "Propositiones and acuendos iuvenes." Although circulated around the year 1000, Alcuin of York (735–804) is said to have authored these as he referred to them in a letter to his most famous student, Charlemagne. The solution given by these works is to carry over the goat, then transport the wolf and return with the goat, then carry over the cabbage, then carry over the goat. A second solution, which simply interchanges the wolf and cabbage, is often attributed to the French mathematician Chuquet in 1484 but is found even earlier in the twelfth century in Germany in the succinct form of Latin hexameter [**1**, **4**, **5**, **23**].

The exact authorship is less important than the fact that the problem was circulated for hundreds of years both orally and in writing. The puzzle is repeated over and over again in mathematical recreation books [**1**, **2**, **7**, **8**, **11**, **15**, **19**, **25**] and has been found as a folk puzzle by collectors of folklore. Sometimes, however, the characters are changed so that a sheep replaces the goat or the trio are replaced by a fox, a fowl, and some corn. Among others, the puzzle is noted in collections of Gaelic, Danish, Russian, Italian, Rumanian, and Black American folklore [**6**, **12**, **20**, **3**, **21**, **9**]. It is one component of a lengthy tale, collected in the French Brittany region [**17**, pp. 208–217], about Jean L'Hébété (the dazed or simple minded) who is eventually given more intelligence by a good fairy in exchange for his wife's solution to her river crossing predicament.

Despite their differences, all of these share the same logical structure: A, B, C must be transported across a river in a boat that can only hold the human rower and one of A, B, or C; neither A nor C can be left alone with B on either shore.

3. African versions Puzzles with the same logical structure are found in Africa among the Tigre (Ethiopia) [**14**, p. 40], on the Cape Verde Islands [**18**] and among the

Bamileke (Cameroon) [**16**]. In the latter, the water is only a stream but the tiger, sheep, and a big spray of reeds have to be walked across individually on the trunk of a fallen tree.

Other related but different problems occur in three regions in Africa. They are similar in that they require a human to transport across a river a predator, its prey, and some food. However, closer examination shows that *they have a distinctly different logical structure*. Now A, B, and C must be transported across a river by a human who can only transport *two* of A, B, C at one time. Neither A nor C can be left alone with B on either shore. What is more, while they superficially share this structure, they contain conscious variations on it.

The most straightforward statement of this is found among the Kabjlie (Algeria) [**10**, p. 246]. In it a man must cross the river with a jackal, goat, and bundle of hay. His solution is to take the jackal and goat, leave the jackal while returning with the goat and then carry across the goat and hay. But, the story continues, another traveler, seeing this, comments that this solution is less efficient than carrying over the jackal and hay and returning for the goat because the goat is being carried on all the trips. "Or", he adds, "did you think that jackal eat hay?" Thus, in the exchange, the traveler points out that a *good* solution should be concerned not only with the *number* of trips but with the lightest load on each trip. And, furthermore, he notes that because the jackal cannot be alone with the goat and the goat cannot be alone with the hay does not imply that the jackal cannot be alone with the hay. In this version of the story there is no boat; the river is sufficiently shallow so that the man can walk across carrying one of the objects under each arm. At first glance this does not seem to affect the logical structure of the problem. But comparison with a Kpelle version shows that it does indeed.

The Kpelle problem is part of a lengthy story [**24**, pp. 445–448]. Set in the northern part of the Kpelle regions (Liberia), the story tells of a king who has a caged cheetah that grabs and eats any fowl near it. The king challenges a suitor for his daughter's hand to transport the cheetah, a fowl, and some rice across the river in a boat that holds one person and two of these. But the king points out, the man cannot control them while rowing the boat, and so in addition to the cheetah and fowl or fowl and rice not being alone together on either shore, they also cannot be together on the boat. The Kpelle problem, therefore, has the additional constraint that neither A nor C can be in the boat with B. The young man tries various solutions and has to appeal to his father for replacement of the fowl and rice when the solutions fail. Eventually he succeeds by carrying over the cheetah and rice and returning for the fowl. Thus, here too, alternative solutions are examined with some ruled out because they involve the cheetah and fowl or the fowl and rice in the boat at the same time.

There is also a Swahili version of the problem. It too is part of a lengthy story of trials [**13**, pp. 19–20, 77–78]. Collected at the turn of the century, the story is set in a sultanate such as was found in Zanzibar until the late 1800's. When a visitor from another region refuses to pay tribute to the sultan, he is confronted with a challenge. If he can carry a leopard, a goat, and some tree leaves to the sultan's son, tribute will be given him and he can remain in the sultanate. The price of failure will be death. The son, of course, lives across a river and the available boat can only hold the traveler and two of the items. The problem, as stated by the traveler, is that he cannot leave *any* two things together on a shore alone. Thus, his solution is that of the Kabjlie with the leaves and leopard interchanged, that is to carry first the leaves and goat, return with the goat, and then carry over the goat and leopard.

These three African problems clearly form a logical unit with slightly differing constraints so that different solutions are appropriate. The logical structure of the basic problem combined with the elaborate stories and the recognition of multiple solutions distinguish these from the predator/prey/food river-crossing problem described for the West.

Still one more African version of the problem is found only among the Ila (Zambia) [**22**, p. 333]. The striking difference is that it involves four items to be transported: a leopard, a goat, a rat, and a basket of corn. The boat can hold just the man and one of these. This problem exemplifies the interrelationship of culture and logical constraints. After considering leaving behind the rat or leopard (and thus reducing the problem to one that can be solved logically), the man's decision is that since both animals are to him as children, he will forego the river crossing and remain where he is!

4. Conclusion The differences in logical structure suggest that the Western and African versions of the problem were independently conceived. Similarity of puzzle goal is not sufficient to imply historical connection. Although the situation depicted seems fanciful if viewed from a twentieth-century, industrial urban setting, the need to get unmanageable items across some water is not uncommon today in other settings and surely was not uncommon during the last thousand years. Even if the puzzles were historically related, the importance is that each group made the problem its own. The case presented here focuses on logic. It is but one example of many that demonstrate that mathematical ideas are of concern in traditional non-Western cultures as well as in the West.

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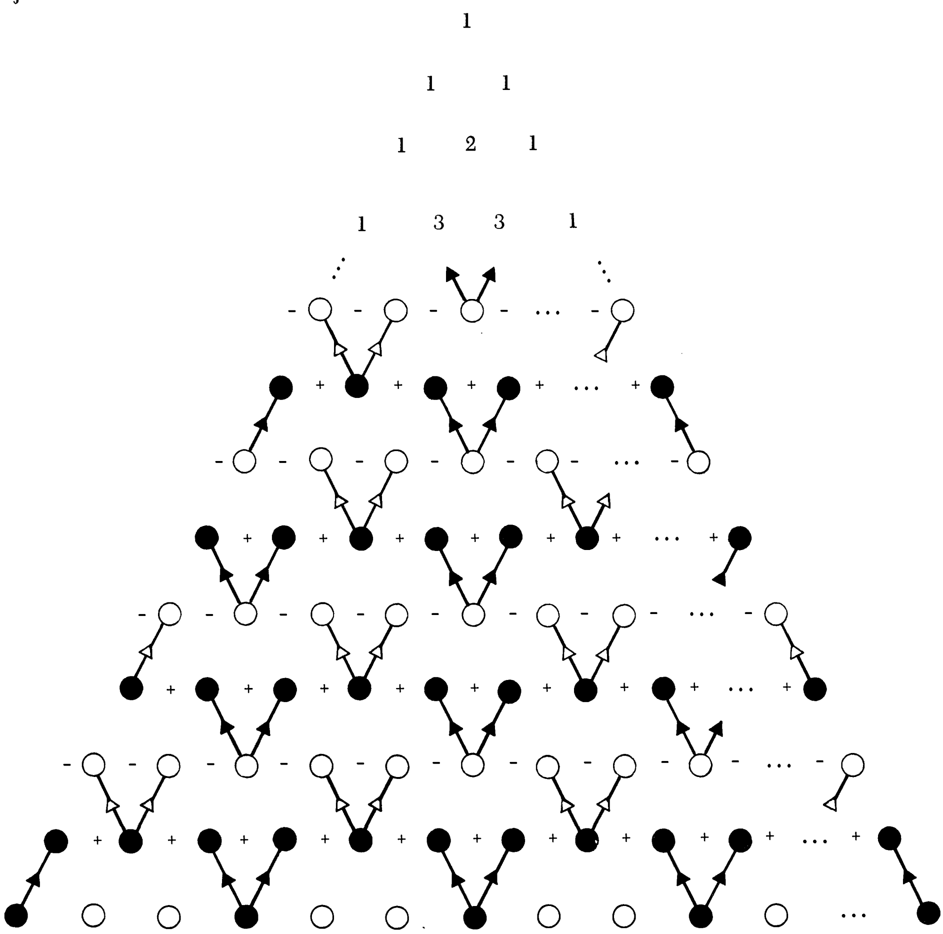
Proof without Words: $3\sum_{j=0}^{n}\binom{3n}{3j} = 8^n + 2(-1)^n$, by Inclusion-Exclusion in Pascal's Triangle



$$\sum_{j=0}^{n}\binom{3n}{3j} = \sum_{j=1}^{3n-1}(-1)^{j-1}2^{3n-j} = -2^{3n}\sum_{j=1}^{3n-1}\left(-\frac{1}{2}\right)^{j} = \frac{8^n+2(-1)^n}{3}.$$

—DEAN S. CLARK
University of Rhode Island
Kingston, RI 02881

On the Rational Solutions of $x^y = y^x$

MARTA SVED
The University of Adelaide
Adelaide, Australia 5001

The number 16 has the interesting property that it can be written as a power of a positive integer in two ways:

$$16 = 2^4 = 4^2.$$

The question arises naturally: Are there any other pairs of integers (x, y) such that

$$x^y = y^x? \tag{1}$$

The problem is not new and not too difficult, but it does not appear in standard texts, so it is not widely known.

In a letter by Daniel Bernoulli to Goldbach (1728) [1], equation (1) is mentioned with the statement that $(x, y) = (2,4)$ (or $(4,2)$) is the only integer solution. In his answer Goldbach gives the general solution of (1) by writing

$$y = ax,$$

hence, $x^{ax} = (ax)^x$ and after simplification, and ignoring the trivial case when $a = 1$,

$$x = a^{1/(a-1)} \quad \text{and} \quad y = a^{a/(a-1)}. \tag{2}$$

Equation (1) is also discussed in some detail by Euler in [2].

The graph of the implicit function $y^x - x^y = 0, (x > 0, y > 0)$ is given as shown in FIGURE 1.

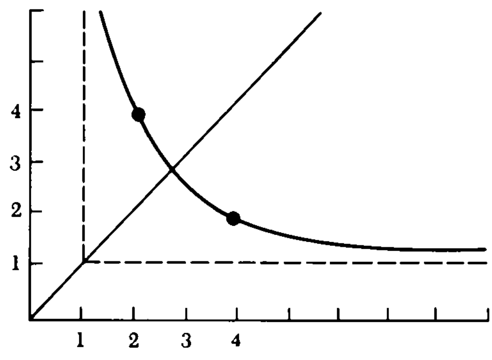


FIGURE 1

Setting $u = 1/(a-1)$ in (2), obtain

$$x = \left(1 + \frac{1}{u}\right)^u \quad \text{and} \quad y = \left(1 + \frac{1}{u}\right)^{u+1}. \tag{3}$$

Thus the graph consists of two branches: The case $a = 1$ yields the line $x = y$, while the parametric equation (3) is represented by the curve symmetrical about the above line and having $x = 1$ and $y = 1$ as asymptotes (as $u \to 0^+$ and $u \to 0^-$, respectively). The intersection of the curve and the line represents

$$\left(\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{u},\ \lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{u+1}\right)=(e,e).$$

The only lattice points on the curved branch are (2,4) and (4,2), but setting integer values for u in (3), Euler obtains an infinite number of isolated points with rational coordinates:

$$\left(\frac{9}{4},\frac{27}{8}\right),\left(\frac{64}{27},\frac{256}{81}\right)\dots$$

It seems that around the turn of the century equation (1) was quite fashionable. L. E. Dickson, in his *History of the Theory of Numbers* [3], cites a number of contributions prior to 1951. Noteworthy and easily accessible is a discussion of the complete locus for real x and y by E. J. Moulton [4].

The problem surfaced again in 1960 as a Putnam Competition question asking for the integer solutions of (1). This prompted A. Hausner to extend results to algebraic number fields [5].

The approach to be described in the present note is aimed at finding *all the nontrivial rational solutions* of (1). It will be shown that these are given by a sequence

$$(a_n,b_n)\to(e,e),$$

where a_n and b_n turn out to be identical to the expressions (3) found by Goldbach and Euler, where positive integers are substituted into (3).

Discarding the trivial solution $x=y$ of (1), we assume that $y>x$, and write (1) in the form

$$y=x^{y/x},$$

or dividing by x, obtain

$$x^{(y/x)-1}=\frac{y}{x}. \tag{4}$$

Set

$$\frac{y}{x}-1=\frac{m}{n}, \tag{5}$$

where m, n are positive integers. Furthermore we assume that m/n is reduced, hence the greatest common divisor

$$(m,n)=1.$$

Equation (4) becomes

$$x^{m/n}=\frac{m+n}{n}$$

or

$$x=(m+n)^{n/m}/n^{n/m}. \tag{6}$$

Since m and n are coprime, so are $m+n$ and n, hence also

$$((m+n)^n,n^n)=1.$$

It follows then from (6) that x is rational if and only if both $(m+n)^n$ and n^n are mth powers.

This implies that each of $m+n$ and n must be an mth power, since m and n (treated now as exponents) are coprime.

Hence

$$n = a^m$$

and

$$m + n = b^m,$$

where a, b are positive integers and $b > a$.

This is *possible if and only if* $m = 1$, since the difference between two consecutive mth powers is $> m$, if $m > 1$.

So by (6)

$$x = \left(1 + \frac{1}{n}\right)^n$$

and from (4)

$$y = \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{for } n = 1, 2, \ldots .$$

In particular, *the only integer solution* of (1) when $y > x$, is obtained when $n = 1$, giving the result

$$x = 2, \qquad y = 4.$$

It is interesting to look graphically at the above results. We want pairs (x, y) for which

$$x^{1/x} = y^{1/y} \tag{7}$$

or, equivalently,

$$\frac{1}{x}\ln x = \frac{1}{y}\ln y. \tag{8}$$

From the graph of $f(t) = \frac{1}{t}\ln t$ $(t > 0)$ (FIGURE 2) it is easy to see that for all $t > e$ there is exactly one value t', where $1 < t' < e$ such that $f(t') = f(t)$.

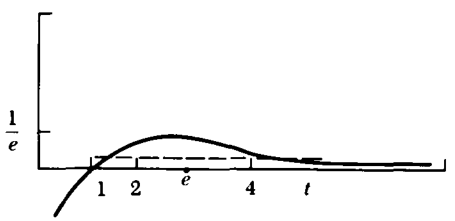


FIGURE 2

Thus a bijective map exists between the *points* on the graph for $t > e$ and pairs (x, y) satisfying (7). The points corresponding *to rational pairs* are *all* on the curve *over the interval* $e < t \leqslant 4$, an illustration of the "scarcity" of rational solutions.

The author thanks the referee for drawing her attention to the history of the problem.

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Wronskian Harmony

MOGENS ESROM LARSEN

Københavns Universitets Matematiske Institut
Universitetsparken 5
DK-2100 København, Denmark

Recently J. Wichmann in Osnabrück considered the Wronski-type determinants for the harmonic oscillators $\sin x, \sin 2x, \dots, \sin kx$. He found these determinants for $k \leqq 7$ to be a coefficient times a power of $\sin x$, and hence suggested the formula

$$f_n(x) = \det\left(\frac{d^j(\sin(k+1)x)}{dx^j}\right)_{j,k=0,\dots,n-1} = K_n(\sin x)^{S_n}, \tag{1}$$

with $S_n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ and the coefficients K_n to be explored, e.g.

$$\begin{aligned}
f_1(x) &= \sin x\\
f_2(x) &= -2\sin^3 x\\
f_3(x) &= -16\sin^6 x\\
f_4(x) &= 768\sin^{10} x\\
f_5(x) &= 294912\sin^{15} x\\
f_6(x) &= -1132462080\sin^{21} x\\
f_7(x) &= -52183852646400\sin^{28} x\\
&\text{etc.}
\end{aligned}$$

The elusive formula cannot be completely trivial since it must depend on properties which the sine does not share with the cosine. The corresponding determinants with "cos" substituted for "sin" give several terms, e.g.,

$$\begin{vmatrix} \cos x & \cos 2x \\ -\sin x & -2\sin 2x \end{vmatrix} = 2\sin^3 x - 3\sin x$$

$$\begin{vmatrix} \cos x & \cos 2x & \cos 3x \\ -\sin x & -2\sin 2x & -3\sin 3x \\ -\cos x & -4\cos 2x & -9\cos 3x \end{vmatrix} = 16\cos x\sin^5 x - 40\cos x\sin^3 x.$$

On the other hand, if we substitute e^x for $\sin x$, then we obtain a formula similar to

(1), but this time the coefficients are easy to compute:

$$\det\left(\frac{d^j e^{(k+1)x}}{dx^j}\right)_{j,k=0,\dots,n-1} = L_n(e^x)^{S_n}, \tag{2}$$

where S_n is the same as in (1) and $L_n = \prod_{j=0}^{n-1} j!$

A table of L_n for $n \leqslant 7$ is

TABLE 1.

n	1	2	3	4	5	6	7
L_n	1	1	2	12	288	34560	24883200

As the reader might have noticed, L_n is a divisor of K_n for $n \leqslant 7$. Furthermore, the quotient K_n/L_n is a power of -2, to be precise,

$$K_n = L_n \cdot (-2)^{S_{n-1}}. \tag{3}$$

The aim of this note is to prove the beautiful formula (1) with K_n defined by (3). We shall begin with the formula (2).

Proof of (2). In each column of the matrix the function is always the same, e.g., $e^{(k+1)x}$. Taking these factors outside we get the formula (2) with coefficient

$$L_n = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 4 & 9 & 16 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 2^{n-1} & 3^{n-1} & 4^{n-1} & \cdots & n^{n-1} \end{vmatrix}. \tag{4}$$

This is the well-known Vandermonde determinant and is hence equal to:

$$\prod_{1 \leqslant j < k \leqslant n} (k-j) = \prod_{k=2}^{n} \prod_{j=1}^{k-1} (k-j) = \prod_{k=2}^{n} (k-1)! = \prod_{j=1}^{n-1} j! \tag{5}$$

However, it turns out that neither the formula (2) nor its proof shall be needed to show (1) and (3).

The proof of (1) and (3) depends on two lemmata to be proved later. The first lemma is due to Chebysheff:

LEMMA 1. *The trigonometric functions* $\sin nx$ *and* $\cos nx$ *can be expressed as polynomials in* $\sin x$ *and* $\cos x$ *as follows*:

$$\cos nx = 2^{n-1}(\cos x)^n + \cdots + (-1)^{n/2} \tag{6}$$

$$\sin nx = \sin x\left(2^{n-1}(\cos x)^{n-1} + \cdots\right) \tag{7}$$

with the term $(-1)^{n/2}$ *only to be included for n even.*

The polynomials are the Chebysheff polynomials.

LEMMA 2. *The Wronski determinants satisfy*

$$\det\left(\frac{d^j(fg_k)}{dx^j}\right)_{j,k=0,\dots,n-1} = f^n \det\left(\frac{d^j g_k}{dx^j}\right)_{j,k=1,\dots,n-1} \tag{8}$$

for $g_0 = 1$ *and* $f, g_1, \ldots, g_{n-1}$ *any functions.*

The difference between sine and cosine is disclosed by lemma 1 as the difference between the polynomials (6) and (7).

Proof of (1) and (3). In the matrix in (1) we write each function as a sum of polynomials as in (7).

$$f_n(x) = \begin{vmatrix} \sin x & \sin x(2\cos x) & \sin x(4\cos^2 x - 1) & \cdots & \sin x(2^{n-1}\cos^{n-1} x + \cdots) \\ \vdots & & \text{“derivatives”} & & \end{vmatrix}.$$

Starting in the third column, we notice that the term " $-\sin x$" gives a repetition of the first column. Hence it can be omitted. The following lower order terms must suffer the same fate as this one as we proceed from left to right. Hence, we are left with the highest powers of $\cos x$.

$$f_n(x) = \begin{vmatrix} \sin x & \sin x(2\cos x) & \sin x(4\cos^2 x) & \cdots & \sin x(2^{n-1}\cos^{n-1} x) \\ \vdots & & \dfrac{d^j(\sin x 2^k \cos^k x)}{dx^j} & & \\ & & & & j, k = 0, \ldots, n-1 \end{vmatrix}.$$

On this form lemma 2 applies. From (8) we get

$$f_n(x) = \sin^n x \cdot \det\left(\frac{d^j(2^k \cos^k x)}{dx^j}\right)_{j,k=1,\ldots,n-1} \tag{9}$$

The first row in the matrix of (9) is ($k = 1, \ldots, n-1$):

$$\frac{d(2^k \cos^k x)}{dx} = 2^k \cdot k \cdot (\cos x)^{k-1} \cdot (-\sin x) = -2k \cdot \left(\sin x(2^{k-1}\cos^{k-1} x)\right). \tag{10}$$

This row is similar to the first row above except for the factor $-2k$. Taking this factor outside, we obtain the recursion formula:

$$f_n(x) = \prod_{k=1}^{n-1} (-2k)\sin^n x f_{n-1}(x) = (-2)^{n-1}(n-1)!\sin^n x f_{n-1}(x). \tag{11}$$

From this follows, with $f_0(x) = 1$,

$$f_n(x) = \prod_{k=1}^{n} (-2)^{k-1}(k-1)!\sin^k x = (-2)^{S_{n-1}} L_n (\sin x)^{S_n} = K_n (\sin x)^{S_n}. \tag{12}$$

This proof could not work for cosines in the obvious analogy. But including $1 = \cos 0x = (\cos x)^0$ it works. We have

$$g_n(x) = \det\left(\frac{d^j \cos(kx)}{dx^j}\right)_{j,k=0,\ldots,n} = 2^{S_{n-1}} L_{n+1}(-\sin x)^{S_n}. \tag{13}$$

Proof. Straightforward computation gives

$$g_n(x) = (-1)^n n! f_n(x).$$

Proof of lemma 1. The Chebysheff polynomials (6) are widely used, but the corresponding polynomials for $\sin nx$ (7) are less known. An easy computation gives both at once:

$$\cos nx + i\sin nx = (\cos x + i\sin x)^n = \sum_{j=0}^{n} \binom{n}{j} (\cos x)^{n-j} (i\sin x)^j$$

$$= \sum_{\substack{j=0\\ j\text{ even}}}^{n} (-1)^{j/2} \binom{n}{j} (\cos x)^{n-j} (1-\cos^2 x)^{j/2} +$$

$$i\sin x \sum_{\substack{j=0\\ j\text{ odd}}}^{n} (-1)^{(j-1)/2} \binom{n}{j} (\cos x)^{n-j} (1-\cos^2 x)^{(j-1)/2}$$

$$= \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{2k} (\cos x)^{n-2k} \sum_{m=0}^{k} (-1)^m (\cos x)^{2m} \binom{k}{m} +$$

$$i\sin x \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1} (\cos x)^{n-2k-1} \sum_{m=0}^{k} (-1)^m (\cos x)^{2m} \binom{k}{m}$$

$$= \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} \cos^n x + \cdots + (-1)^{n/2} + i\sin x \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} \cos^{n-1} x + \cdots$$

$$= 2^{n-1}\cos^n x + \cdots + (-1)^{n/2} + i\sin x 2^{n-1}\cos^{n-1} x + \cdots$$

where the term $(-1)^{n/2}$ only occurs for n even, and the dots stands for terms of lower order.

Proof of lemma 2. If $f(x)=0$, then everything is zero. So we can assume $f(x)\neq 0$. Leibniz' formula for derivation is

$$\frac{d^j(fg)}{dx^j} = \sum_{k=0}^{j} \binom{j}{k} f^{(k)} g^{(j-k)}. \tag{14}$$

Using (14) row by row in the matrix we find that the last terms in (14) give us a part proportional to the first row.

$$\frac{f^{(k)}}{f} \cdot (f, fg_1, \ldots, fg_{n-1}) = \left(f^{(k)}, f^{(k)}g_1, \ldots, f^{(k)}g_{n-1}\right).$$

So these terms can be subtracted from each row except the first one. Then the determinant looks like this:

$$\begin{vmatrix} f & fg_1 & fg_2 & \cdots & fg_{n-1} \\ 0 & fg_1' & fg_2' & \cdots & fg_{n-1}' \\ \vdots & & \displaystyle\sum_{k=0}^{j-1} \binom{j}{k} f^{(k)} g_m^{(j-k)} & & \\ 0 & & & & \end{vmatrix}.$$

Now, the second to last term of the sum, i.e.,

$$\binom{j}{j-1} f^{(j-1)} g'_m$$

gives a part proportional to the second row as it looks now. Hence this part can be omitted from the third row and on. Continuing this way we end up with the determinant looking like

$$\begin{vmatrix} f & fg_1 & fg_2 & fg_3 & \cdots & fg_{n-1} \\ 0 & fg'_1 & fg'_2 & fg'_3 & \cdots & fg'_{n-1} \\ 0 & fg''_1 & fg''_2 & fg''_3 & & fg''_{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & fg_1^{(n-1)} & & & & fg_{n-1}^{(n-1)} \end{vmatrix}.$$

Taking the factor f outside and developing the determinant after the first column one gets lemma 2 immediately.

Residual Lifetimes in Random Parallel Systems

FREDERICK SOLOMON
SUNY at Purchase
Purchase, NY 10577

You go to the Statue of Liberty and return with a dozen souvenir glasses. They begin to break soon thereafter, but the last glass in the set seems to linger on for years. Why is it that residual 'components' in such a random system last a surprisingly long time? In the autumn leaves drop from the tree in the front yard; most of the leaves are clustered close to the trunk, but a few come to rest much further distant than one might expect. Neutrons in a reactor are absorbed by atoms in the shielding material; bullets shot into a forest are stopped by trees. How are the residual neutrons/bullets —the ones that are stopped most distant from the source—distributed?

This note explores the distributions of such residual 'components'; in particular, why does the last component last so long (or outdistance all the others by so much)? Two models are considered in detail. The first consists of a system of components with exponentially distributed lifetimes; here the residual components are those with greatest lifetimes. Starting with a large number N at time 0 equations (11), (12), and (13) show how the expected number of surviving components C depends on certain 'benchmark' times; most notable is equation (11), which is an approximation to the number of surviving components at fraction α of the total time until the final component dies. Equation (21) is a recursive formula for the fraction of total system lifetime during which only one component is living. Equation (14) is analogous to (11), but applies to the two-dimensional 'leaf model' in which objects fall on a plane with positions independent and normally distributed.

Definition A *parallel system of N components* consists of N components which all begin operating at time 0. The components are labeled from 1 to N and the ith component has a lifetime T_i which is random. The random variables $T_1, \dots, T_N$ are independent, each with the same distribution function

$$F(t)=P(T_i\leqslant t),$$

where $P(\cdot)$ denotes probability. In fact, formulas for general F are derived (subject to the conditions stated below), but the integrals can only be evaluated in two special cases; these are the exponential distribution and another which is closely related to it. For clarity, however, we'll use a general notation. In deriving the formulas (until the actual substitution of a particular distribution), we assume the following properties of F: F is continuous on $(-\infty,\infty)$, $F(0)=0$, $F(t)\to 1$ as $t\to\infty$, and F has a continuous derivative on $(0,\infty)$: The density function for each of the N components is

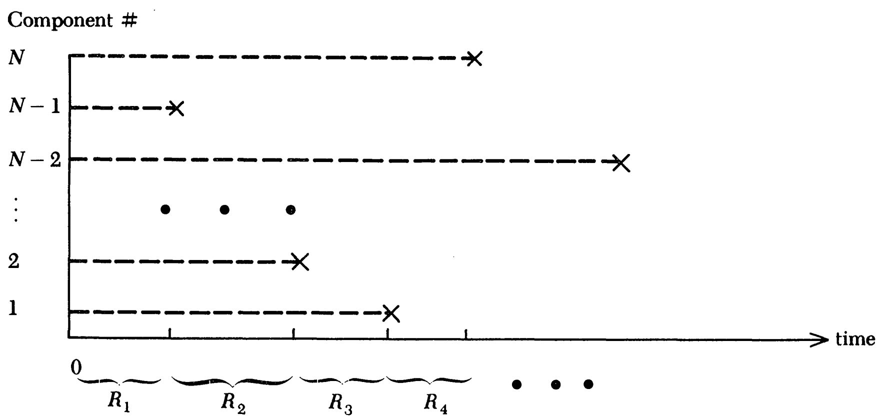
$$f(t)=F'(t),$$

for $0<t<\infty$. For the *exponential distribution*

$$F(t)=1-e^{-\lambda t}$$
$$f(t)=\lambda e^{-\lambda t},$$

for $0<t<\infty$ where λ is a constant $\lambda>0$.

A typical plot of lifetimes is



The times $R_1,\ldots,R_N$ are the successive times between component deaths or failures. That is, suppose that the shortest-lived component dies at time t_1 and the next component to die does so at time t_2. Then $R_1=t_1$ and $R_2=t_2-t_1$ *regardless* of which components die first and second. Thus R_i is the time between the $(i-1)$st component failure and the ith component failure. Thus

$$S(i)=R_1+\cdots+R_i$$

is the time of the ith component failure. In particular the total system lifetime is $S(N)$ which is the time till *all* the components have died. Let D_t denote the number of component deaths in $[0,t)$. (It will turn out to be simpler in the formulas to take the interval open at the right end t.) Since the component distribution function F is continuous, the distribution for the number of component deaths in $[0,t]$ is the same as the distribution for the number of component deaths in $[0,t)$. Let $C_t=N-D_t$ be the number of survivors at time $t-0$. There is some subtlety of notation here. For a *fixed* time t, the number of survivors at time t is equal to the number of survivors at time $t-0$ with probability 1. For a *random* time the situation can differ. Thus in time $[0,S(1)]$ exactly one component dies, but in time $[0,S(1))$ no component dies. Therefore $D_{S(1)}=0$ and $C_{S(1)}=N$. Similarly, the number of component deaths in

$[0, S(i)]$ is i, but the number of component deaths in $[0, S(i))$ is $i-1$. Thus $D_{S(i)} = i-1$ and $C_{S(i)} = N - D_{S(i)} = N-(i-1) = N-i+1$. Note that $C_t \to N$ as $t \searrow 0$.

The probability that component i dies in the time interval $[0,t]$ is $F(t)$; thus independence of the component lifetimes implies that D_t is binomially distributed with $p = F(t)$. Hence the expected number of deaths in $[0,t)$ is

$$E(D_t) = N \cdot F(t)$$

This agrees with intuition: The expected *fraction* of the total number of components that have died by time t is $E(D_t)/N$ which is $F(t)$ = probability that any individual component has died by time t. But this formula doesn't explain why the *last few* components seem to linger on long after the others have died.

The *Lack of Memory* property for the individual component lifetime T_i states that for times $t, s \geqslant 0$

$$P(T_i > t+s \mid T_i > t) = P(T_i > s),$$

where $P(A|B)$ denotes the conditional probability of A given B. That is, if it is known that the component has lasted till time t, then its future lifetime distribution is as though it were brand new. Certainly this would be the case for each glass in a set; its breakage depends on conditions not determined by its age. It is a standard problem to show that the Lack of Memory property is equivalent to stating that T_i is exponentially distributed (e.g., [2, page 192]): There is a constant $\lambda > 0$ so that

$$P(T_i > t) = e^{-\lambda t}$$

for $t \geqslant 0$. The expected lifetime is then

$$E(T_i) = 1/\lambda.$$

When the component lifetimes are exponentially distributed with parameter λ, the expected system lifetime $E(S(N))$ satisfies

$$E(S(N)) \cong \frac{1}{\lambda}\ln(N)$$

in the sense that the ratio of the two sides tends to 1 as $N \to \infty$. This is a standard exercise, but for completeness we outline a proof that uses Lack of Memory: Fact 1: The *minimum* of k independent random variables, each exponentially distributed with parameter λ is itself exponentially distributed with parameter $k\lambda$. Fact 2: At time $S(j-1)$ when the $(j-1)$st component dies, the remaining $N-(j-1) = N-j+1$ components have lifetimes *from that instant* which are exponentially distributed with parameter λ because of the Lack of Memory property. (That is, at time $S(j-1)$ each of the remaining $N-j+1$ components behaves as though it is brand new.) Since R_j is the successive time until the next failure and is therefore the minimum failure time of $N-j+1$ exponentially distributed random variables, Fact 1 implies that R_j is exponentially distributed with parameter $(N-j+1)\lambda$ and therefore expectation $1/(N-j+1)\lambda$. Thus

$$\begin{aligned} E(S(N)) &= E(R_1) + \cdots + E(R_N) \\ &= \frac{1}{N\lambda} + \frac{1}{(N-1)\lambda} + \cdots + \frac{1}{\lambda} \\ &= \frac{1}{\lambda} \cdot \sum_{j=1}^{N} \frac{1}{j} \cong \frac{1}{\lambda}\ln(N). \end{aligned}$$

The residual lifetime of the last component to fail after all the others have failed is R_N which has expectation $1/(N-N+1)\lambda = 1/\lambda$. To gain some insight into why the last of a set of glasses lasts so long consider the ratio of the expected residual lifetime to the expected system lifetime:

$$E(R_N)/E(S(N)) \cong 1/\ln(N). \tag{1}$$

N	$1/\ln(N)$
10	.43
100	.22
1000	.14
10000	.11

Since $\ln(N)$ is a very slowly growing function, $E(R_N)/E(S(N))$ approaches 0 as $N \to \infty$, but the rate at which it does so is much slower than intuition might suggest.

Thus we now have *one* answer to the question of why the last component in the system with exponentially distributed lifetimes lasts so long. In the next sections we derive other results showing the relative longevity of the residual components.

Probability and expectation formulas for D_t and C_t Recall that the system consists of N independent components each with distribution function F. Thus D_t is binomially distributed as the number of successes in N Bernoulli trials with $p = F(t)$. Let h_i be the density function for the time $S(i)$ of the ith component death. Then h_i is concentrated on $(0, \infty)$.

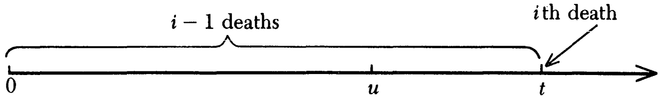
$$\begin{aligned} P(S(i) \leqslant t) &= P(\text{at least } i \text{ deaths in } [0,t]) \\ &= \sum_{k=i}^{N} P(D_t = k) \\ &= \sum_{k=i}^{N} \binom{N}{k} F(t)^k (1-F(t))^{N-k}. \end{aligned}$$

Taking the derivative of this distribution function for $S(i)$ we obtain

$$\begin{aligned} h_i(t) = \sum_{k=i}^{N} \binom{N}{k} \Big[& kf(t)F(t)^{k-1}(1-F(t))^{N-k} \\ & - F(t)^k (N-k) f(t) (1-F(t))^{N-k-1} \Big] \\ = \sum_{k=i}^{N} & \binom{N}{k} kf(t)F(t)^{k-1}(1-F(t))^{N-k} \\ - \sum_{k=i+1}^{N+1} & \binom{N}{k-1} F(t)^{k-1}(N-k+1)f(t)(1-F(t))^{N-k} \\ = N & \binom{N-1}{i-1} F(t)^{i-1}(1-F(t))^{N-i} f(t). \end{aligned} \tag{2}$$

The second equality follows by splitting the sum in the right-hand side of the first equality into two sums; then the index in the second sum is changed from k to $k+1$. The final equality holds because nearly all the addends in the two sums of the right-hand side of the second equality cancel.

Let $0 < u < t$ be fixed times and $0 \leqslant j \leqslant i-1$. Given that the ith death occurred at time t, what is the probability that j deaths occurred by the previous time u?



If $S(i)=t$, then $i-1$ components have died in $[0,t)$. *Given* that a component has died in $[0,t)$, the probability that it has done so in $[0,u)$ is $F(u)/F(t)$. Since the components have independent lifetimes, the number of deaths in $[0,u)$ *given* that there have been $i-1$ deaths in $[0,t)$ is binomially distributed as the number of successes in $i-1$ Bernoulli trials with $p=F(u)/F(t)$. That is,

$$P(D_u=j|S(i)=t)=\binom{i-1}{j}\left(\frac{F(u)}{F(t)}\right)^j\left(1-\frac{F(u)}{F(t)}\right)^{i-1-j} \tag{3}$$

Thus

$$E(D_u|S(i)=t)=(i-1)\cdot p=(i-1)\cdot\frac{F(u)}{F(t)}. \tag{4}$$

Fix a number α with $0<\alpha<1$. The time $\alpha\cdot S(i)$ is the fraction α of the time till the ith death. In particular, $\alpha\cdot S(N)$ is the fraction of the total lifetime of the system. We use (4) to compute the expected number of survivors at time $\alpha\cdot S(i)$.

$$\begin{aligned}E(D_{\alpha S(i)})&=\int_0^\infty E(D_{\alpha t}|S(i)=t)h_i(t)\,dt\\&=\int_0^\infty(i-1)\frac{F(\alpha t)}{F(t)}N\binom{N-1}{i-1}F(t)^{i-1}(1-F(t))^{N-i}f(t)\,dt\\&=N(N-1)\binom{N-2}{i-2}\int_0^\infty F(\alpha t)F(t)^{i-2}(1-F(t))^{N-i}f(t)\,dt.\end{aligned} \tag{5}$$

Applying an integration by parts to (5) for the case $i=N$ we obtain

$$\begin{aligned}E(D_{\alpha S(N)})&=N(N-1)\int_0^\infty F(\alpha t)F(t)^{N-2}f(t)\,dt\\&=N(N-1)\int_0^\infty(1-(1-F(\alpha t)))F(t)^{N-2}f(t)\,dt\\&=N(N-1)\left[\int_0^\infty F(t)^{N-2}f(t)\,dt-\int_0^\infty(1-F(\alpha t))F(t)^{N-2}f(t)\,dt\right]\\&=N\int_0^\infty d[F(t)]^{N-1}-N\int_0^\infty(1-F(\alpha t))d[F(t)]^{N-1}\\&=N\cdot F(t)^{N-1}\Big|_0^\infty-N\left[F(t)^{N-1}(1-F(\alpha t))\Big|_0^\infty+\alpha\int_0^\infty F(t)^{N-1}f(\alpha t)\,dt\right]\\&=N-N\alpha\int_0^\infty F(t)^{N-1}f(\alpha t)\,dt.\end{aligned} \tag{6}$$

In terms of the number of survivors

$$E(C_{\alpha S(N)})=E(N-D_{\alpha S(N)})=N\alpha\int_0^\infty F(t)^{N-1}f(\alpha t)\,dt. \tag{7}$$

Applications Suppose each component lifetime is exponentially distributed with parameter λ. Equation (5) implies

$$\begin{aligned}
E(D_{\alpha S(i)}) &= N(N-1)\binom{N-2}{i-2}\int_0^\infty (1-e^{-\lambda\alpha t})(1-e^{-\lambda t})^{i-2}\cdot e^{-\lambda(N-i)t}\lambda e^{-\lambda t}\,dt \\
&= N(N-1)\binom{N-2}{i-2}\int_0^1 (1-u^\alpha)(1-u)^{i-2}u^{N-i}\,du \\
&= N(N-1)\binom{N-2}{i-2}\left[\int_0^1 (1-u)^{i-2}u^{N-i}\,du - \int_0^1 (1-u)^{i-2}u^{N+\alpha-i}\,du\right] \\
&= N(N-1)\binom{N-2}{i-2}\left[\frac{\Gamma(i-1)\Gamma(N-i+1)}{\Gamma(N)} - \frac{\Gamma(i-1)\Gamma(N+\alpha-i+1)}{\Gamma(N+\alpha)}\right] \\
&= N - \frac{N!}{(N-i)!}\cdot\frac{\Gamma(N+\alpha-i+1)}{\Gamma(N+\alpha)}, \qquad (8)
\end{aligned}$$

where $\Gamma(\cdot)$ denotes the gamma function. The second equality follows by the substitution $u = e^{-\lambda t}$. For the second to last equality see [**1**, pages 621–625]. Therefore

$$E(C_{\alpha S(i)}) = \frac{N!}{(N-i)!}\cdot\frac{\Gamma(N+\alpha-i+1)}{\Gamma(N+\alpha)}. \qquad (9)$$

The case $i = N$ is particularly interesting; we apply Stirling's Formula to approximate the gamma function [**1**, page 635] for large N:

$$\begin{aligned}
E(C_{\alpha S(N)}) &= N!\frac{\Gamma(\alpha+1)}{\Gamma(N+\alpha)} \qquad (10) \\
&\cong \frac{\sqrt{2\pi}\,N^{N+(1/2)}\cdot e^{-N}}{\sqrt{2\pi}\,(N+\alpha-1)^{N+\alpha-(1/2)}\cdot e^{-N-\alpha+1}}\cdot\Gamma(\alpha+1) \\
&= \frac{N}{(N+\alpha-1)^\alpha}\cdot\left(\frac{N}{N+\alpha-1}\right)^{N-(1/2)}\cdot\frac{1}{e^{-\alpha+1}}\cdot\Gamma(\alpha+1) \\
&\cong \frac{N}{(N+\alpha-1)^\alpha}\cdot\Gamma(\alpha+1) \\
&\cong N^{1-\alpha}\cdot\Gamma(\alpha+1), \qquad (11)
\end{aligned}$$

where the approximate equalities are in the sense that the ratio of their two sides tends to 1 as $N \to \infty$. Result (11) is rather interesting: Since $\Gamma(\alpha+1)$ ranges between about .88 and 1 for $0 \leqslant \alpha \leqslant 1$, *the expected number of operational components is of order of magnitude* $N^{1-\alpha}$ *at fraction* α *of the total lifetime* $S(N)$ *of the system.* For example, start with $N = 10{,}000$ glasses. At half the time till the last one finally breaks $(\alpha = 1/2)$ one can expect only $N^{1/2}\cdot\Gamma(3/2) \cong 89$ (less than 1%) unbroken. At three-fourths the time till the last one breaks $(\alpha = 3/4)$ one can expect only $N^{1/4}\cdot\Gamma(7/4) \cong 9$ glasses to be unbroken.

Let $0 < \alpha < 1$ and $0 < \beta < 1$ be fixed. Applying Stirling's Formula to equation (9) in a way similar to that used to derive approximation (11) one can show that

$$E(C_{\alpha S(]\beta N[)}) \cong N\cdot(1-\beta)^\alpha \qquad (12)$$

for large N where $]x[$ denotes the largest integer less than or equal to x. (Again, the approximate equality in (12) is in the sense that the ratio of the two sides tends to 1 as $N \to \infty$.)

On the other hand for small i, the expected number of operational components at time $\alpha S(i)$ is (using result (9))

$$E(C_{\alpha S(i)}) = \frac{N!}{(N-i)!} \cdot \frac{1}{(N+\alpha-1)(N+\alpha-2)\cdots(N+\alpha-i+1)}$$
$$= N \cdot \frac{(N-1)\cdots(N-i+1)}{(N+\alpha-1)\cdots(N+\alpha-i+1)} \tag{13}$$

in which both numerator and denominator contain $i-1$ terms.

Now consider a *two-dimensional model*: Suppose leaves fall from a tree in such a way that their final positions on the ground are independent, each with symmetric *normal density*

$$g(x,y) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{x^2+y^2}{2\sigma^2}\right].$$

(Thus the base of the tree is at $(0,0)$ and the variance for each leaf in each dimension is σ^2.) It is a standard topic in probability texts to show that the *distance squared* from $(0,0)$ to a particular leaf is exponentially distributed with parameter $1/2\sigma^2$ (e.g., see [3, Chap. 10, Sec. 9]. From this we could apply the results from the previous model to this new model. For clarity of exposition, however, it seems preferable to go to the minimal extra effort to derive results for the new 'leaf model' without using the correspondence with the previous model directly. Let $D(t)$ denote the closed disk of radius t centered at $(0,0)$ in R^2. The *distance* from the base of the tree to a particular leaf has distribution function F and density function f given by

$$F(t) = \frac{1}{2\pi\sigma^2}\iint_{D(t)}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)dx\,dy$$
$$= 1-\exp\left(-\frac{t^2}{2\sigma^2}\right)$$

for $t>0$ using polar coordinates for the integration. And

$$f(t) = F'(t) = \frac{t}{\sigma^2}\exp\left(-\frac{t^2}{2\sigma^2}\right)$$

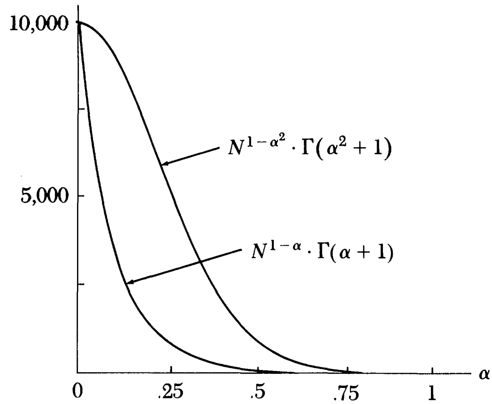
for $t>0$. Applying formula (5) one finds that

$$E(D_{\alpha S(i)}) = N(N-1)\binom{N-2}{i-2}\int_0^\infty (1-e^{-\alpha^2t^2/2\sigma^2})(1-e^{-t^2/2\sigma^2})^{i-2}$$
$$\cdot(e^{-t^2/2\sigma^2})^{N-i}\cdot\frac{t}{\sigma^2}e^{-t^2/2\sigma^2}\,dt$$
$$= N(N-1)\binom{N-2}{i-2}\int_0^1(1-u^{\alpha^2})(1-u)^{i-2}u^{N-i}\,du,$$

using the substitution $u = e^{-t^2/2\sigma^2}$ to obtain the last integral. But notice that this integral is identical to the second integral used to obtain expression (8) except α^2 replaces α! Consequently formulas (8)–(13) hold in this two-dimensional model with α replaced by α^2. In particular

$$E(C_{\alpha S(N)}) \cong N^{1-\alpha^2}\cdot\Gamma(\alpha^2+1) \tag{14}$$

is the analogous result to expression (11). Let us compare (11) and (14): Consider neutrons penetrating shielding material in a reactor in such a way that the Lack of Memory property holds for the distance traveled before absorption. Then expression (11) is valid for the number that penetrate at least fraction α of the total distance to the most distantly absorbed neutron. On the other hand (14) is valid for the two-dimensional leaf model. See the figure below for $N = 10{,}000$.



Thus there is a sharper decrease in surviving components/leaves in the one-dimensional exponential model.

Computation of $E(R_N/S(N))$ in the exponential case The Lack of Memory property implies that $R_1, \ldots, R_N$ are independent when the component lifetimes are exponentially distributed. This fact allows computation of $E(R_N/S(N))$ = expected fraction of the total system lifetime that the last component lasts *after* all the others have died. (Recall that equation (1) describes the behavior of $E(R_N)/E(S(N))$ for N large.) Throughout this section we assume that the components are exponentially distributed with parameter λ. Note that

$$S(N) = S(N-1) + R_N$$

is an independent sum. The density for $S(N-1)$ is h_{N-1}

$$h_{N-1}(t) = N(N-1)F(t)^{N-2}(1-F(t))f(t)$$

from expression (2). Also, Lack of Memory implies that the density for R_N is

$$f(t) = \lambda e^{-\lambda t}$$

for $t > 0$. Therefore, for $0 < t < 1$

$$\begin{aligned} P(R_N/S(N) > t) &= P(R_N > t(S(N-1) + R_N)) \\ &= P\left(R_N > \frac{t}{1-t} S(N-1)\right) \\ &= \int_0^\infty \int_{\frac{t}{1-t}y}^\infty f(x) h_{N-1}(y)\, dx\, dy \\ &= \int_0^\infty \int_{\frac{t}{1-t}y}^\infty N(N-1)\lambda^2 e^{-\lambda x}(1-e^{-\lambda y})^{N-2} e^{-2\lambda y}\, dx\, dy \end{aligned}$$

$$= N(N-1)\int_0^1 u^{1/(1-t)}\cdot(1-u)^{N-2}\,du$$

$$= N(N-1)\cdot\frac{\Gamma\left(\frac{1}{1-t}+1\right)\cdot\Gamma(N-1)}{\Gamma\left(N+\frac{1}{1-t}\right)}$$

$$= N!\cdot\frac{1}{\left(N-1+\frac{1}{1-t}\right)\cdots\left(1+\frac{1}{1-t}\right)}. \tag{15}$$

The fifth equality follows upon integration with respect to x and subsequent substitution of $u = e^{-\lambda y}$. Since the random variable $R_N/S(N)$ is concentrated on $(0,1)$,

$$E(R_N/S(N)) = \int_0^\infty P(R_N/S(N) > t)\,dt$$

$$= N!\int_0^1 \frac{dt}{(N-1+1/(1-t))\cdots(1+1/(1-t))}$$

$$= N!\int_1^\infty \frac{dz}{z^2(z+1)\cdots(z+N-1)}, \tag{16}$$

where the last integral uses the substitution $z = 1/(1-t)$. Using the basic partial fraction technique one obtains

$$E(R_2/S(2)) = 2 - 2\ln(2) = .6137$$
$$E(R_3/S(3)) = 3 - 6\ln(2) + 1.5\ln(3) = .4890$$
$$E(R_4/S(4)) = 4 - (4/3)\ln(2) + 6\ln(3) = .4255.$$

But this quickly becomes unwieldy. A recursion formula can be derived as follows: Set

$$E_N = \frac{1}{N!}E(R_N/S(N)).$$

Then

$$E_N = \int_1^\infty \frac{dz}{z^2(z+1)\cdots(z+N-1)}$$

$$= \int_1^\infty\left[\frac{N}{z^2(z+1)\cdots(z+N)} + \frac{1}{z(z+1)\cdots(z+N)}\right]dz$$

$$= NE_{N+1} + \int_1^\infty \frac{dz}{z(z+1)\cdots(z+N)}. \tag{17}$$

The following lemma is used to find the integral in (17):

LEMMA.

$$\frac{1}{z(z+1)\cdots(z+N)} = \frac{1}{N!}\left[\frac{\binom{N}{0}}{z} - \frac{\binom{N}{1}}{z+1} + \cdots \pm \frac{\binom{N}{N}}{z+N}\right]$$

$$= \frac{1}{N!}\cdot\sum_{j=0}^{N}(-1)^j\binom{N}{j}\frac{1}{z+j}. \tag{18}$$

Proof. Fix z. The right-hand side of (18) is

$$\frac{1}{N!}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\int_0^1 x^{z+j-1}\,dx=\frac{1}{N!}\int_0^1 x^{z-1}\cdot\sum_{j=0}^{N}\binom{N}{j}(-1)^j x^j\,dx$$
$$=\frac{1}{N!}\int_0^1(1-x)^N x^{z-1}\,dx$$
$$=\frac{1}{N!}\cdot\frac{\Gamma(N+1)\Gamma(z)}{\Gamma(z+N+1)}$$
$$=\frac{\Gamma(z)}{\Gamma(z+N+1)},$$

which agrees with the left-hand side of (18).

Using (18) to evaluate the integral in (17) we see that

$$E_N=NE_{N+1}+\frac{1}{N!}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\int_1^\infty\frac{dz}{z+j}$$
$$=NE_{N+1}+\frac{1}{N!}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\ln(z+j)\Big|_1^\infty. \qquad (19)$$

At the upper limit the sum in (19) is

$$\lim_{L\to\infty}\frac{1}{N!}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\ln(L+j)$$
$$=\lim_{L\to\infty}\frac{1}{N!}\ln\left[\frac{L^{\binom{N}{0}}(L+2)^{\binom{N}{2}}\cdots}{(L+1)^{\binom{N}{1}}(L+3)^{\binom{N}{3}}\cdots}\right]$$
$$=\frac{1}{N!}\ln\left(\lim_{L\to\infty}\left[\frac{L^{\binom{N}{0}}(L+2)^{\binom{N}{2}}\cdots}{(L+1)^{\binom{N}{1}}(L+3)^{\binom{N}{3}}\cdots}\right]\right).$$

The log of the limit is 0 since the largest exponent of L in both numerator and denominator is the same; that is, the largest exponent of L in the numerator is

$$\binom{N}{0}+\binom{N}{2}+\cdots=\sum_{\substack{j=0\\ j\text{ even}}}^{N}\binom{N}{j}=2^{N-1}.$$

And the largest exponent of L in the denominator is

$$\binom{N}{1}+\binom{N}{3}+\cdots=\sum_{\substack{j=1\\ j\text{ odd}}}^{N}\binom{N}{j}=2^{N-1}.$$

Consequently (19) implies that

$$E_N=NE_{N+1}-\frac{1}{N!}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\ln(j+1). \qquad (20)$$

Thus the definition of E_N yields this recursion relation:

$$E(R_{N+1}/S(N+1)) = \frac{N+1}{N}E(R_N/S(N)) + \frac{N+1}{N}\cdot\sum_{j=0}^{N}(-1)^j\binom{N}{j}\ln(j+1) \tag{21}$$

for $N = 1, 2, \ldots$ where

$$E(R_1/S(1)) = E(1) = 1.$$

N	$E(R_N/S(N))$
2	.61
5	.38
10	.30
15	.26
20	.20

This again shows why the last in a parallel system seems to last so long.

Distributions of R_j and a general consequence For $j = 1$

$$\begin{aligned} P(R_1 > u) &= P(T_i > u, \quad \text{for } i = 1, \ldots, N) \\ &= (1 - F(t))^N. \end{aligned}$$

For $j \geqslant 2$

$$\begin{aligned} P(R_j > u) &= \int_0^\infty P(R_j > u \mid S(j-1) = t) h_{j-1}(t)\, dt \\ &= \int_0^\infty P(\text{remaining } N-j+1 \text{ survive till } t+u \mid j-1\text{st dies at } t)\, h_{j-1}(t)\, dt \\ &= \int_0^\infty \left(\frac{1-F(t+u)}{1-F(t)}\right)^{N-j+1} \cdot N\binom{N-1}{j-2}F(t)^{j-2}(1-F(t))^{N-j+1}f(t)\, dt \\ &= N\binom{N-1}{j-2}\int_0^\infty (1-F(t+u))^{N-j+1}F(t)^{j-2}f(t)\, dt. \end{aligned} \tag{22}$$

An integration by parts yields

$$\begin{aligned} P(R_j > u) &= N\binom{N-1}{j-2}\left[\frac{1}{j-1}(1-F(t+u))^{N-j+1}F(t)^{j-1}\Big|_0^\infty \right. \\ &\qquad \left. + \frac{N-j+1}{j-1}\int_0^\infty (1-F(t+u))^{N-j}F(t)^{j-1}f(t+u)\, dt\right] \\ &= N\binom{N-1}{j-1}\int_0^\infty (1-F(t+u))^{N-j}F(t)^{j-1}f(t+u)\, dt \end{aligned} \tag{23}$$

for $2 \leqslant j \leqslant N$. In fact (23) is also valid for $j = 1$ since in that case the right-hand side of (23) evaluates to $(1 - F(t))^N$ which is $P(R_1 > u)$. (One can easily check this formula in the exponential case in which $f(t) = \lambda e^{-\lambda t}$; then $P(R_j > u) = e^{-(N-j+1)\lambda t}$.) Notice that if in (22) j were replaced by $j+1$, then the resulting expression would be expression (23) *except* that $f(t)$ in the integrand in (22) would not match up with $f(t+u)$ in the integrand of (23). A consequence of this fact is the following:

THEOREM. *Suppose that f is strictly decreasing on $(0, \infty)$. Then*

$$P(R_j > u) < P(R_{j+1} > u)$$

for all $u > 0$. Therefore, $E(R_j) < E(R_{j+1})$.

The Theorem states this: After each 'component' dies, one can expect to wait longer for the next death than one expected to wait for the previous one.

A specific formula for $E(R_j)$ can be obtained from (23):

$$\begin{aligned} E(R_j) &= \int_0^\infty P(R_j > u)\,du \\ &= N\binom{N-1}{j-1}\int_0^\infty\int_0^\infty (1-F(t+u))^{N-j}F(t)^{j-1}f(t+u)\,dt\,du \\ &= N\binom{N-1}{j-1}\int_0^\infty\left[\frac{-1}{N-j+1}(1-F(t+u))^{N-j+1}\right]_{u=0}^{\infty}F(t)^{j-1}\,dt \\ &= \binom{N}{j-1}\int_0^\infty F(t)^{j-1}(1-F(t))^{N-j+1}\,dt. \end{aligned}$$

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Demystifying the Projective Plane

MICHAEL H. BRILL
Science Applications International Corp.
1710 Goodridge Drive T-11
McLean, VA 22102

MICHAEL STUEBEN
4651 Brentleigh Ct.
Annandale, VA 22003

It is standard [1, 2] to describe the projective plane topologically as one of three surfaces formed from a square by joining each pair of opposite edges in a definite way. FIGURE 1 shows the three possibilities; in each case, opposite pairs of edges are to be joined so the arrows coincide. If each part of FIGURE 1 were a video screen, a point x moving off the right-hand edge of the screen would reappear at the left as x', and a point y moving off the top of the screen would reappear at the bottom as y'. In the standard treatments, the torus and Klein bottle—but not the projective plane—are given clear intuitive interpretations. However, even lucid reviews that seem otherwise

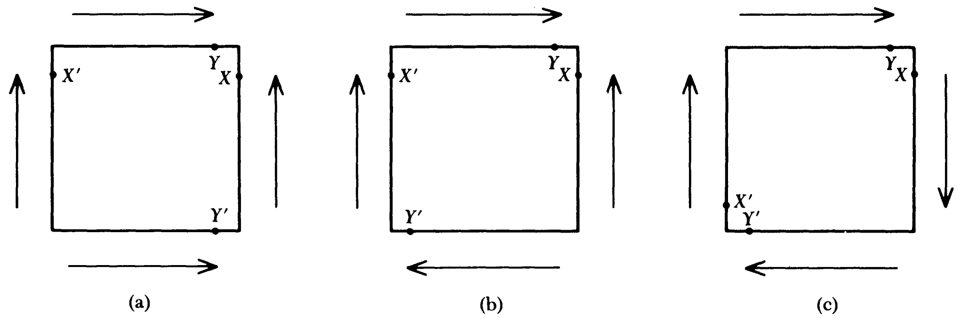

FIGURE 1
Three topological surfaces: (a) torus; (b) Klein bottle; (c) projective plane.

complete [3, 4, 5] do not explicitly say what the name "projective plane" has to do with FIGURE 1c. We would like to explain the connection in this note.

The projective plane represents the geometry of central projection familiar in perspective drawing and in photography (see FIGURE 2). In central projection, points in three-dimensional space are mapped along straight lines passing through a common focal point F to an image plane P. We will show that certain properties of the projection of a moving point to P are represented by FIGURE 1c. As a point A moves continuously in three dimensions, the image point a on P also moves continuously, until A crosses the plane S (called the focal plane) that passes through F parallel to P. Notice that, as A moves across plane S, the image a jumps out to infinity, and then reappears from the other side of the image plane. The direction from which a reappears is simply the negative of the direction in which a rushes to infinity as A crosses plane S. This is precisely the rule represented in FIGURE 1c.

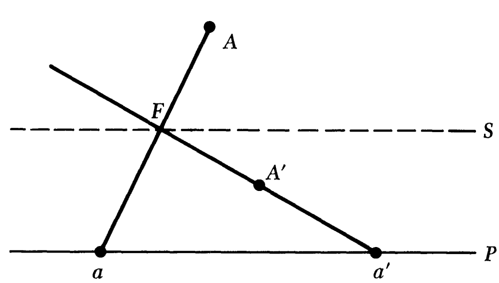

FIGURE 2
Perspective imaging in projective plane P. Shown are the respective images a, a' of points A, A' on opposite sides of focal (or separatrix) plane S.

Perspective drawing derives its rules from the properties of central projection, some of which we state here and then prove in the next paragraph. The projective mapping to the image plane P from the half space on either side of S is many-to-one, hence near objects occlude far objects in a perspective drawing. The image of a straight line L that neither is parallel to P nor contains F is a straight line with a single point missing (see FIGURE 3). The missing point is approached arbitrarily closely in the image as the point on L approaches infinity on either side of S. In perspective drawing, the missing point is called a *vanishing point*, and is shared by the images of all lines parallel to L (except for the image of the line L_0 containing F, which *is* the

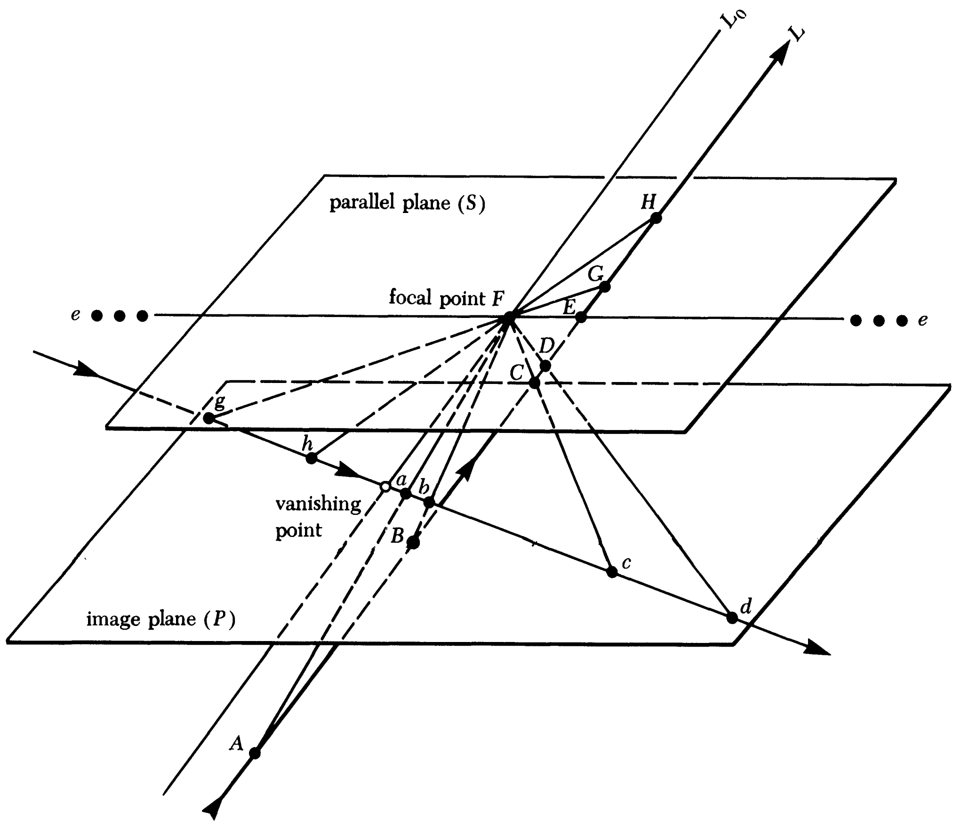

FIGURE 3
Perspective image of a line L. Images of the parts of the line in the two half-spaces are separated by the missing, or vanishing point. The image of a parallel line L_0 is the vanishing point of L.

vanishing point). A line parallel to P maps to a *complete* line in P, with no vanishing point. Finally, although the passage of L through P is an anomaly for physical imaging, there is no mathematical discontinuity in the mapping at the crossing point; but there *is* a discontinuity where L crosses S.

Some remarks in proof: That line L maps to another line under central projection can be seen by noting the straight-line intersection of P with the plane passing through L and F. That parallel lines L_k have the same vanishing point can be seen by expressing a point on L_k as $\boldsymbol{X}_k = \boldsymbol{Q}_k + t\boldsymbol{R}$, where t is a parameter and, e.g., vector $\boldsymbol{R} = (R_x, R_y, R_z)$. Choose P as the locus $Z = 1$, and F as the point $(0,0,0)$. Then the image of $\boldsymbol{X}_k$ is

$$\boldsymbol{x}_k = \left(\frac{Q_{kx} + tR_x}{Q_{kz} + tR_z}, \frac{Q_{ky} + tR_y}{Q_{kz} + tR_z}, 1 \right).$$

As $t \to \infty$, $\boldsymbol{X}_k \to \boldsymbol{R}/R_z$, which is the vanishing point and is independent of k. The anomalous cases described in the last paragraph appear as $R_z = 0$ (L_k parallel to P) and $\boldsymbol{Q}_k = 0$ (L_k passes through F).

The well known fact that the projective plane cannot be oriented emerges from the fact that the right/left handedness of the image of three noncollinear points can change when the points are translated across plane S. In particular, the handedness reverses when the translation is in the plane defined by the points (assuming this plane is neither parallel to P nor passing through F). Because the half-spaces in front of and behind S map separately and with opposite orientations (left/right handedness) to the image plane, orientation is not an intrinsic property of the points under translation. In the video-screen analogy (FIGURE 1c), which conveys topological but not other properties of central projection to P, the wrap-around rule can be seen to reverse the clockwise/counterclockwise ordering of three points as they pass across any edge of the screen.

In attempting to connect FIGURE 1c into a closed surface subject to its wrap-around rules, one approach [1, 2] is to stretch the square $ABCD$ of FIGURE 4a (adapted from FIGURE 1c) until it becomes a spherical shell with four edges (FIGURE 4b). The edges are then sewn together to form a seam, with A meeting C and B meeting D (FIGURE 4c). The wrap-around rule can then be traced by looking back at FIGURE 4b. Note that crossing the seam in FIGURE 4c entails moving from AB to CD (or the reverse), or from BC to DA (or the reverse). The other transitions that look possible at the seam are not allowed. Hence the construction is called a "cross-cap." From the previous paragraph, it is then clear that an asymmetrical figure confined to the surface changes its handedness when moved across the seam, and can then be superimposed on a mirror image of itself that did not cross the seam. Analogous remarks, of course, apply to the Klein-bottle surface of FIGURE 1b.

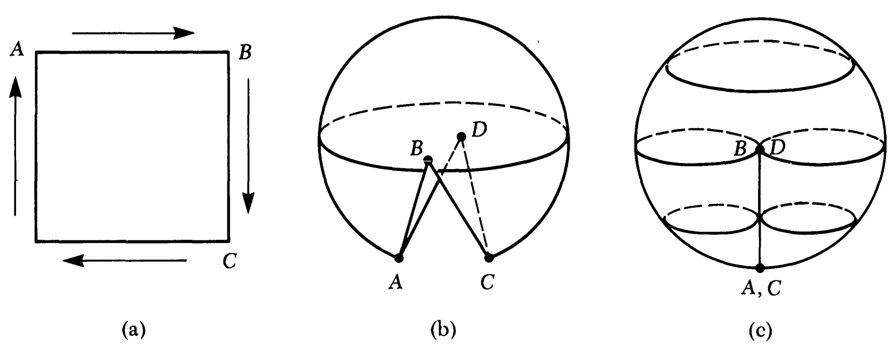

FIGURE 4
Closed-surface representation of the projective plane: (a) original square; (b) distorted square; (c) cross-cap.

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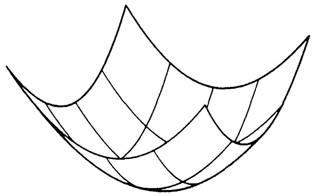
5. H. B. Griffiths and P. J. Hilton, *A Comprehensive Textbook of Classical Mathematics: A Contemporary Interpretation*, Van Nostrand, 1970, Ch. 17.

Definitely ~ Positively the Pits

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ positive definite

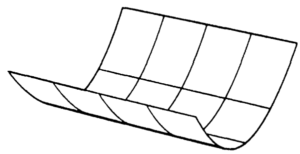
$$z = [x, y]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2$$



pit

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ positive semidefinite

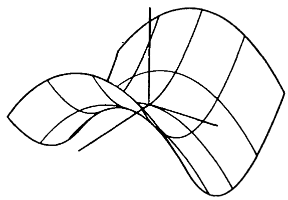
$$z = [x, y]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 0$$



valley

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ indefinite

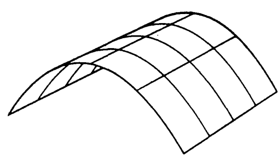
$$z = [x, y]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 - y^2$$



pass

$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ negative semidefinite

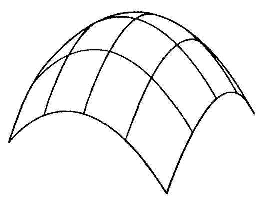
$$z = [x, y]\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 - y^2$$



ridge

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ negative definite

$$z = [x, y]\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - y^2$$

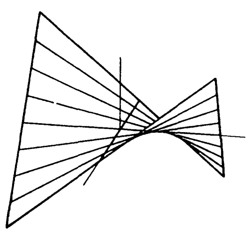


peak

Rotate to Saddle-up

$$z = 2xy = [x, y]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x, y]\begin{bmatrix} a & -a \\ a & +a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & a \\ -a & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = s^2 - t^2;$$

where $a = \sqrt{2}/2$



—Cliff Long
Bowling Green St. U.
Bowling Green, OH 43403

Generating Nonpowers by Formula

A. J. DOS REIS
SUNY
New Paltz, NY 12561

D. M. SILBERGER*
SUNY
New Paltz, NY 12561

For x a real number the expression $[x]$ as usual denotes the greatest integer $t \leqslant x$. We note that $[x+0.5]$ is the integer closest to x except when x is equidistant from two integers. The symbol N denotes the set of all positive integers.

For each integer $n>1$ we define the function f_n: $N \to N$ by $f_n(i) =$ the ith positive integer which is not a perfect nth power. For example in the table below we see the behavior of f_2; namely, f_2 lists in ascending order the elements in N which are not perfect squares.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	...
$f_2(i)$	2	3	5	6	7	8	10	11	12	13	14	15	17	18	19	20	21	22	23	24	26	...

On pages 97 and 98 of [**1**], an explicit formula

$$f_2(i) = i + \left[\sqrt{i} + 0.5\right] \quad \text{for every } i \in N \tag{1}$$

is derived for f_2, and on pages 98 and 99 a quite different formula

$$f_2(i) = i + \left[\sqrt{i + [\sqrt{i}\,]}\,\right] \quad \text{for every } i \in N \tag{2}$$

is also established. The formula (1) is mentioned in [**2**]. In the present paper, we establish for each integer $n>1$ an explicit formula for f_n.

Whereas we have no interest in the sequence $\langle f_n(1), f_n(2), f_n(3), \dots \rangle$ *per se*, inasmuch as this sequence can be obtained by deleting the perfect nth powers from the sequence $\langle 1, 2, 3, \dots \rangle$, the idea of a simple formula capable of generating these sequences $\langle f_n(i) \rangle$ is an intriguing one. Indeed, we find it remarkable that a single simple formula using only the operations of addition, subtraction, multiplication, division, powers, roots, and the greatest integer function can skip over exactly those integers necessary to realize such a sequence.

LEMMA. *Let $n>1$ be an integer. Then*

$$f_n(k^n) = k^n + k. \tag{3}$$

Proof. Among the integers in the set $Q = \{1, 2, \dots, k^n + k\}$, there are exactly k perfect nth powers: $1^n, 2^n, \dots, k^n$. Clearly, there are exactly $(k^n + k) - k = k^n$ non-nth-powers in Q, and thus $f_n(k^n) = k^n + k$.

THEOREM. *For each pair $n>1$ and $i>0$ of integers, let $p(n,i)$ denote $([i^{1/n}] + 1)^n - [i^{1/n}]$. Then*

*On leave from Universidade Federal de Santa Catarina, 88.000-Florianópolis-SC-Brasil.

$$f_n(i) = i + [i^{1/n}] + [i/p(n,i)] \quad \textit{for each } i \in N. \tag{4}$$

Proof. Choose $i \in N$. Then there is a unique $k \in N$ for which $k^n \leqslant i < (k+1)^n$. So $k \leqslant i^{1/n} < k+1$, and thus $k = [i^{1/n}]$. Also it is clear that $p(n,i) = (k+1)^n - k > 0$.

Since the positive integer $(k+1)^n - 1$ is not a perfect nth power, there exists $j_k \in N$ such that $f_n(j_k) = (k+1)^n - 1$. Since the function f_n is strictly increasing, and since obviously $k^n + k < (k+1)^n - 1 < (k+1)^n + (k+1)$—*i.e.*, $f_n(k^n) < f_n(j_k) < f_n((k+1)^n)$—it follows from (3) that $k^n < j_k < (k+1)^n$. Moreover, $f_n(j_k + 1) = (k+1)^n + 1$ since $(k+1)^n$ is a perfect nth power.

Clearly, $k^n < f_n(k^n) < f_n(j_k) = (k+1)^n - 1 < (k+1)^n < f_n(j_k+1) = (k+1)^n + 1 < f_n((k+1)^n) < (k+2)^n$. That is to say, there is exactly one perfect nth power—namely $(k+1)^n$—between $f_n(k^n)$ and $f_n((k+1)^n)$.

In the table below, we see the behavior of $f_n(i)$ for $k^n \leqslant i \leqslant (k+1)^n$.

i	k^n	k^n+1	...	j_k	j_k+1	...	$(k+1)^n$
$f_n(i)$	k^n+k	k^n+k+1	...	$(k+1)^n-1$	$(k+1)^n+1$	...	$(k+1)^n+k+1$

Note that as the argument i increases consecutively from k^n to $(k+1)^n$, the values $f_n(i)$ also increase consecutively except that $f_n(i)$ skips $(k+1)^n$. Thus for $k^n \leqslant s \leqslant t \leqslant (k+1)^n$

$$f_n(t) - f_n(s) = \begin{cases} t - s + 1 & \text{if } s \leqslant j_k \text{ and } t > j_k \\ t - s & \text{otherwise.} \end{cases} \tag{5}$$

By (5) we have that $f_n((k+1)^n) - f_n(j_k) = (k+1)^n - j_k + 1$. Then by substituting for $f_n((k+1)^n)$ and $f_n(j_k)$ and solving for j_k, we get

$$\begin{aligned} j_k &= (k+1)^n - k - 1 \\ &= \left([i^{1/n}] + 1\right)^n - [i^{1/n}] - 1 \\ &= p(n,i) - 1. \end{aligned} \tag{6}$$

Again using (5), we have that $f_n(i) = f_n(k^n) + i - k^n$ if $k^n \leqslant i \leqslant j_k$. Substituting for $f_n(k^n)$, we get that $f_n(i) = (k^n + k) + i - k^n = i + k$ and thus that

$$f_n(i) = i + [i^{1/n}] \quad \text{if } k^n \leqslant i \leqslant j_k. \tag{7}$$

Similarly we have that $f_n(i) = f_n(k^n) + i - k^n + 1$ if $j_k < i < (k+1)^n$. Thus, $f_n(i) = (k^n + k) + i - k^n + 1 = i + k + 1$ and

$$f_n(i) = i + [i^{1/n}] + 1 \quad \text{if } j_k < i < (k+1)^n. \tag{8}$$

By (6) if $k^n \leqslant i \leqslant j_k$ then $k^n \leqslant i < p(n,i)$, whence $0 < i/p(n,i) < 1$. It follows that

$$[i/p(n,i)] = 0 \quad \text{if } k^n \leqslant i \leqslant j_k. \tag{9}$$

Suppose on the other hand that $j_k < i < (k+1)^n$. Then $p(n,i) \leqslant i < (k+1)^n$ whence $1 \leqslant i/p(n,i) < (k+1)^n/p(n,i) = (k+1)^n/((k+1)^n - k)$. But since $n > 1$ and since $k > 0$ we see that $2k < (k+1)^2 \leqslant (k+1)^n$, and hence that $k < (k+1)^n/2$. Therefore, $i/p(n,i) < (k+1)^n/((k+1)^n - k) < (k+1)^n/((k+1)^n - (k+1)^n/2) = 2$. It follows that $1 \leqslant i/p(n,i) < 2$, and, hence, that

$$[i/p(n,i)] = 1 \quad \text{if } j_k < i < (k+1)^n \tag{10}$$

The theorem is now the immediate consequence of (7) and (8) together with (9) and (10).

Define a function to be *elementary* if it can be expressed by an explicit finite formula using only the operations of addition, subtraction, multiplication, division, roots, powers, and the greatest integer function. Define an injective sequence $\langle v_1, v_2, v_3, \dots \rangle$ to be *elementarily generated* if there exists an elementary function f such that $f(i) = v_i$ for every $i = 1, 2, 3, \dots$. By the *complement of a sequence* $t = \langle t_1, t_2, t_3, \dots \rangle$ of positive integers we mean the sequence obtained from $s = \langle 1, 2, 3, \dots \rangle$ by deleting from s all of the terms in t.

An easy cardinality argument shows that there exist subsequences of s which are not elementarily generated. For, while there are only countably many elementary functions there are uncountably many subsequences of s.

We believe the following questions to be open:

(i) For positive integers a and n, if $f(x) = ax^n$, then is the complement of $\langle f(1), f(2), f(3), \dots \rangle$ elementarily generated? Our theorem answers this question affirmatively for the cases where $a = 1$ and $n > 1$.

(ii) If $p(x)$ is a polynomial of positive degree with nonnegative integer coefficients, then is the complement of $\langle p(1), p(2), p(3), \dots \rangle$ elementarily generated?

(iii) Is each finite increasing sequence of positive integers elementarily generated?

(iv) Is the complement of a finite sequence of positive integers elementarily generated?

(v) Is the complement of an elementarily generated sequence of positive integers elementarily generated?

Our work has profited from the assistance of David M. Clark, who moreover provided the proof of (3) used in this paper. M. W. Ecker was helpful in directing us to the literature.

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2. M. W. Ecker, Mathematical recreations: the fundamental counting principle, *Byte Magazine* 10 (1985), 425–428.

A Pessimistic Note on Fermat's Last Theorem

$$(3 + \sqrt{93})^3 + (3 - \sqrt{93})^3 = 12^3.$$

—JAMES P. CONLAN
Menlo College
Atherton, CA 94025

PROBLEMS

LOREN C. LARSON, *editor*
St. Olaf College

GEORGE GILBERT, *associate editor*
St. Olaf College

Proposals

To be considered for publication, solutions should be received by July 1, 1990.

1338. *Proposed by Andrew Cusumano, Great Neck, New York.*

Evaluate

$$F(k) \equiv \lim_{n\to\infty} n^k\left[\left(1+\frac{1}{n+1}\right)^{n+1}-\left(1+\frac{1}{n}\right)^{n}\right].$$

1339. *Proposed by George Gilbert, St. Olaf College, Northfield, Minnesota.*

Find all integer triples (x,y,z), $2 \le x \le y \le z$, such that

$$xy \equiv 1 \pmod{z}$$
$$xz \equiv 1 \pmod{y}$$
$$yz \equiv 1 \pmod{x}.$$

1340. *Proposed by Richard L. Francis, Southeast Missouri State University, Cape Girardeau, Missouri.*

Show that the 3-degree angle is the only constructible angle of prime degree measure.

ASSISTANT EDITORS: CLIFTON CORZAT and THEODORE VESSEY, *St. Olaf College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren C. Larson, Department of Mathematics, St. Olaf College, Northfield, MN 55057.

1341. *Proposed by Bernardo Recamán, Universidad de los Andes, Bogotá, Colombia.*

a. Determine all positive integers n for which it is always possible to label the vertices of a planar graph on n vertices with the first n positive integers, so that adjacent vertices will have labels that are relatively prime.

b*. Is it always possible to label the vertices of a planar graph on n vertices with the first n *odd* positive integers, so that adjacent vertices will have labels that are relatively prime?

1342. *Proposed by Greg Fredricks and Harvey Schmidt, Jr., Lewis and Clark College, Portland, Oregon.*

Let $\mathbf{A} = (a_{i,j})$ be the upper triangular $n \times n$ matrix where

$$a_{i,j} = (-1)^{i+j}\binom{n-i+1}{j-1}$$

for $i \leq j$. Find the inverse of $\mathbf{A}$.

Quickies

Answers to the Quickies are on p. 63.

Q757. *Proposed by Jozsef Mala and Sandor Roka, Bessenyei College, Nyiregyhaza, Hungary.*

Does there exist a continuous real-valued function on $\mathbf{R}$ whose values are rational on the irrationals and irrational on the rationals?

Q758. *Proposed by Murray S. Klamkin, University of Alberta, Alberta, Canada.*

Evaluate

$$I = \int_0^a x^n(2a-x)^n\,dx \div \int_0^a x^n(a-x)^n\,dx.$$

Q759. *Proposed by Norman Schaumberger, Bronx Community College, Bronx, New York.*

If a, b, c, and d are the lengths of the sides of a quadrilateral and if P is its perimeter, then

$$\frac{abc}{d^2} + \frac{bcd}{a^2} + \frac{cda}{b^2} + \frac{dab}{c^2} > P$$

unless $a = b = c = d$.

Solutions

A Deranged Sequence **February 1989**

1312. *Proposed by Daniel Ulman, George Washington University, Washington, D.C.*

For $n \geq 1$, let $S_n = \Sigma 1/a_1 a_1 \cdots a_m$, where the sum is taken over all m and all finite sequences of positive integers $a_1, a_2, \ldots, a_m$, such that $a_1 = n$ and $a_{i+1} \leq a_i - 2$ for $1 \leq i \leq m-1$. For example,

$$S_6 = \frac{1}{6} + \frac{1}{6\cdot 4} + \frac{1}{6\cdot 3} + \frac{1}{6\cdot 2} + \frac{1}{6\cdot 1} + \frac{1}{6\cdot 4\cdot 2} + \frac{1}{6\cdot 4\cdot 1} + \frac{1}{6\cdot 3\cdot 1}.$$

Show that S_n is a convergent sequence and find its limit.

I. *Solution by J. S. Frame, Michigan State University, East Lansing, Michigan.*

We note that $S_1 = 1$, and define $S_0 = 0$. Then for $n \geq 2$

$$nS_n = 1 + S_1 + S_2 + \cdots + S_{n-2},$$
$$nS_n - (n-1)S_{n-1} = S_{n-2}.$$

Hence, using induction on n, we have

$$S_n - S_{n-1} = -(S_{n-1} - S_{n-2})/n = (-1)^{n-1}(S_1 - S_0)/n! = (-1)^{n-1}/n!$$

Thus S_n is the nth partial sum of the convergent alternating series whose sum is $1 - e^{-1}$, the required limit of S_n.

II. *Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, New York.*

Consider the generating function $S(x) = \Sigma_{n \geq 1} S_n x^n$. We have $S_1 = 1$, $S_2 = 1/2$, and for $n \geq 3$, $nS_n - (n-1)S_{n-1} = S_{n-2}$. Multiplying each side by x^n and summing yields

$$x \sum_{n\geq 3} nS_n x^{n-1} - x \sum_{n\geq 3} (n-1)S_{n-1}x^{n-2} = x^2 \sum_{n\geq 3} S_{n-2}x^{n-2},$$

which is equivalent to

$$S'(x) - 1 - x - x(S'(x) - 1) = xS(x).$$

Hence, we have to solve the differential equation $(1-x)S'(x) - xS(x) = 1$, subject to the initial condition $S(0) = 0$. The solution is

$$\frac{1-e^{-x}}{1-x} = \left(\sum_{n=0}^{\infty} x^n\right)\left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!}\right)x^n.$$

Thus, $S_n = 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ and S_n converges to $1 - e^{-1}$.

III. *Solution by Lou Shapiro, Howard University, Washington, D.C.*

Let $[n] = \{1, 2, \ldots, n\}$, let D_n denote the number of permutations of $[n]$ with no fixed points, let F_n be the number of permutations of $[n]$ with at least one fixed point. Then $D_n + F_n = n!$. It is well known that $\lim_{n\to\infty} D_n/n! = e^{-1}$.

Select a permutation of $[n]$ at random, and let P_n denote the probability that it has a fixed point. It is known (e.g., see L. Lovász, *Combinatorial Problems and Exercises*, North Holland, Exercise 3.3) that the probability that the cycle containing 1 has length k is $1/n$ (independent of k). Thus

$$P_n = \frac{1}{n} + \frac{1}{n}\sum_{k=2}^{n-1} P_{n-k}, \qquad P_1 = 1,\ P_2 = 1/2. \tag{1}$$

The first $1/n$ on the right side is the probability that 1 is a fixed point. Otherwise, 1 is in a k-cycle (with probability $1/n$). The elements in the k-cycle with 1 are not fixed points, so the remaining $n-k$ points must provide any fixed points.

But (1) is precisely the recurrence satisfied by S_n as defined in the statement of the problem. Thus, $S_n = P_n = F_n/n! = 1 - D_n/n!$. It follows that S_n converges to $1 - e^{-1}$.

Also solved by Brian D. Beasley, J. C. Binz (Switzerland), W. E. Briggs, Nicholas Buck (Canada), David Callan, Jeffrey Clark, Con Amore Problem Group (Denmark), Robert Doucette, Roger B. Eggleton (Brunei Darusalam), Richard A. Gibbs, G. A. Heuer, Thomas Jager, Benjamin G. Klein, Kenneth A. Klinger, Y. H. Harris Kwong (second solution), Kee-Wai Lau (Hong Kong), Eugene Levine, Hosam M. Mahmoud, David E. Manes, Helen M. Marston, Reiner Martin (student, West Germany), Alasdair McAndrew (Australia), Howard Morris, Stephen Noltie, Teunis J. Ott, Paul B. Peart and Leon Woodson, Jim Pfaendtner, Robert L. Raymond, Rob L. Reid (student), Volkhard Schindler (East Germany), John S. Sumner, Samuel Vandervelde, Michael Vowe (Switzerland), William P. Wardlaw, Richard A. Weida, Don West, Morton Zweiback, A. Zulauf (New Zealand), and the proposer.

Levine *and* Reid *observed that the number of terms in the sum defining S_n is the nth Fibonacci number. Ott solved a slightly more general problem: Let $S_k^{(n)} = \sum 1/a_1a_2\cdots a_m$, where the sum is taken over all finite sequences of positive integers $(a_1, a_2, \ldots, a_m)$ with $a_1 = k$ and $a_{i+1} \le a_i - n$ for $1 \le i \le m-1$. (Hence, $n(m-1) \le k-1$.) Then, for $[x] < 1$,*

$$\sum_{k=1}^{\infty} S_k^{(n)} x^k = \frac{1}{1-x}\int_0^x \exp\left(-\sum_{k=1}^{n-1}\frac{1}{k}(x^k - u^k)\right)du.$$

Nonintuitive Exponentials **February 1989**

1313. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

Let $f(x) = a^x$. For what values of a, $0 < a < 1$, if any, are there distinct points P and Q on the graph of $y = f(x)$ that are symmetrical about the line $y = x$?

Solution by Thomas Jager, Calvin College, Grand Rapids, Michigan.

Distinct points exist if and only if $0 < a < e^{-e}$.

Let λ be the unique number such that $a^\lambda = \lambda$. Then, distinct points exist if and only if there is an $x > \lambda$ such that

$$f(x) \equiv \frac{a^{a^x}}{x} = 1.$$

Now,

$$f'(x) = \frac{a^{a^x}}{x^2}\left(a^x \ln(a^x) \ln a - 1\right).$$

If $a > e^{-e}$ then $-e < \ln a < 0$. Also, $-1/e < y \ln y < 0$ for $0 < y < 1$. Thus, $f'(x) < 0$ for $x > \lambda$. This, together with $f(\lambda) = 1$ implies that there is no $x > \lambda$ such that $f(x) = 1$.

If $a = e^{-e}$, then $\lambda = 1/e$, $f'(\lambda) = 0$, and as above, $f'(x) < 0$ for $x > \lambda$, and we have the same conclusion.

If $0 < a < e^{-e}$, then $\lambda^{1/\lambda} = a < e^{-e} = (e^{-1})^{1/e^{-1}}$. Since $x^{1/x}$ is increasing on (0,1),

$\lambda < e^{-1}$, so that $a = \lambda^{1/\lambda} < e^{-1/\lambda}$. Hence, $a^\lambda \ln a^\lambda \ln a - 1 = \lambda^2(\ln a)^2 > \lambda^2(-1/\lambda)^2 - 1 = 0$. Thus, $f'(\lambda) > 0$. This, together with $f(\lambda) = 1$ and $f(1) = a^a < 1$, implies that there is an $x > \lambda$ such that $f(x) = 1$.

Also solved by Centre College Mathematical Problem Solving Group, Con Amore Problem Group (Denmark), Robert Doucette, Lee O. Hagglund, King Jamison, Benjamin G. Klein, David W. Koster, Lamar University Problem Solving Group, David E. Manes, Fouad Nakhli, Roger B. Nelsen, Stephen Noltie, Harry Sedinger and Charles R. Diminnie, Michael Vowe (Switzerland), William P. Wardlaw, A. Zulauf (New Zealand), and the proposer. There were four incorrect solutions.

Klein *notes that the problem is closely related to the problems treated in* "Exponentials Reiterated", *by R. A. Knoebel, Amer. Math. Monthly* 88 (1981), 235–252. *Also, see Problem* 687, *Crux Mathematicorum* 8 (1982), 305–306 *and G. Klambauer, Problems and Propositions in Analysis, Marcel Dekker, New York,* 1979, *pp.* 186–193.

Sums of Triangular Numbers — February 1989

1314. *Proposed by Jany C. Binz, University of Bern, Switzerland.*

Let $t_n = n(n+1)/2$ be the nth triangular number. Find all positive integers m and n such that

$$t_n + t_{n+1} + t_{n+2} = t_m.$$

Solution by Thomas Jager, Calvin College, Grand Rapids, Michigan.

The solutions are (n_k, m_k), $k = 0,1,\ldots$ where

$$n_k = \frac{1}{2}\left(\frac{1+\sqrt{3}}{2}(2+\sqrt{3})^k + \frac{1-\sqrt{3}}{2}(2-\sqrt{3})^k\right) - \frac{3}{2}$$

$$m_k = \frac{3}{2}\left(\frac{1+\sqrt{3}}{2\sqrt{3}}(2+\sqrt{3})^k - \frac{1-\sqrt{3}}{2\sqrt{3}}(2-\sqrt{3})^k\right) - \frac{1}{2}.$$

The equation given is equivalent to $3(n+1)(n+2) = (m-1)(m+2)$. Hence $3|(m-1)$ and $3|(m+2)$. The change of variables $x = 2n+3$, $y = (2m+1)/3$ produces $(*)x^2 - 3y^2 = -2$, an instance of Pell's equation (all of whose solutions have x and y odd). Consider the transformation T defined on $\mathbf{Z} \times \mathbf{Z}$ by $T(x, y) = (2x + 3y, x + 2y)$. It is easy to show that if (x, y) satisfies $(*)$, then $T(x, y)$ and $T^{-1}(x, y) = (2x - 3y, -x + 2y)$ do as well. Then the positive solutions to $(*)$ are $(x_k, y_k) = T^k(1,1)$ for $k \geq 0$. That all (x, y) are solutions follows from the above.

Suppose there are positive solutions not of this form. Let (a, b) be a minimal such. Then $T^{-1}(a, b)$ must be a non-positive solution; i.e., $2a - 3b \leq 0$ or $-a + 2b \leq 0$. The latter implies that $4b^2 \leq a^2 = 3b^2 - 2$, or $b^2 \leq -2$, an impossibility. The former implies that $9b^2 \geq 4a^2 = 12b^2 - 8$, or $8 \geq 3b^2$. Hence $(a, b) = (1,1)$, contrary to hypothesis. Now T has eigenvalue, eigenvector pairs

$$v_1 = (\sqrt{3}, 1), \quad \lambda_1 = 2 + \sqrt{3}, \quad v_2 = (-\sqrt{3}, 1), \quad \lambda_2 = 2 - \sqrt{3}.$$

Since $(x_0, y_0) = (1,1)$,

$$(x_k, y_k) = T^k(1,1) = \frac{1+\sqrt{3}}{2\sqrt{3}}\lambda_1^k v_1 + \frac{\sqrt{3}-1}{2\sqrt{3}}\lambda_2^k v_2.$$

Then, from $n_k = (x_k - 3)/2$ and $m_k = (3y_k - 1)/2$, we get the result above.

Also solved by S. F. Barger, W. E. Briggs, Duane Broline, David Callan, John M. Coker, Con Amore Problem Group (Denmark), David Doster, Roger B. Eggleton (Brunei Darussalam), Kevin Ford (student), Lorraine L. Foster, Cornelius Groenewoud, Russell Jay Hendel (two solutions), Francis M. Henderson, J. Heuver (Canada), Hans Kappus (Switzerland), Stelios Kapranidis, L. Kuipers (Switzerland), Lamar University Problem Solving Group, Stewart A. Levin, Eugene Levine, David E. Manes, Roger B. Nelsen, F. D. Parker, Alan Pedersen (Denmark), Bob Prielipp, Don Redmond, William Romaine, Jim Rue and Jerry Metzger, Volkhard Schindler (East Germany), Kyle Sherrill and James Foley, Kiran Lall Shrestha (Nepal), Harry D'Souza, Robert Stacy, John S. Sumner, Michael Vowe (Switzerland), Shirley Wakin, Kyle Wallace and Glenn Powers and Robert Crawford, William P. Wardlaw, Western Maryland College Problem Group, A. Zulauf (New Zealand), Morton Zweiback, and the proposer. In addition, there were two partial solutions and three incorrect solutions.

Polynomial Family with Complex Roots **February 1989**

1315. *Proposed by M. S. Klamkin and A. Liu, University of Alberta, Canada.*

Determine all real values of λ such that the roots of

$$P(x) \equiv x^n + \lambda \sum_{r=1}^{n} (-1)^r x^{n-r} = 0, \; (n > 2)$$

are *all* real.

I. *Solution by Seung-Jin Bang, Seoul, Korea.*

If $\lambda = 0$ then all the roots of P are real. We will show that if $\lambda \neq 0$ then P has at least one nonreal root.

Suppose $\lambda \neq 0$, and set $x = -1/y$. The equation becomes

$$G(y) \equiv 1/\lambda + \sum_{r=1}^{n} y^r.$$

Suppose the roots $r_1, r_2, \ldots, r_n$ of P are all real. Then the roots of G are $s_i \equiv 1/r_i$, $i = 1, 2, \ldots, n$, and

$$\sum_{i=1}^{n} s_i^2 = \Big(\sum_i s_i\Big)^2 - 2\sum_{i<j} s_i s_j = 1^2 - 2 = -1,$$

a contradiction.

II. *Solution by David Callan, University of Bridgeport, Bridgeport, Connecticut.*

If $\lambda = 0$ obviously all roots are real. Suppose $\lambda \neq 0$. We can write $P(x)$ as $Q(x)/(x+1)$ where $Q(x) = x^{n+1} + (1-\lambda)x^n + (-1)^n\lambda$. Since $P(0) \neq 0$ the number of real roots of P is the number of positive roots of $P(x)$ plus the number of positive roots of $P(-x)$. By Descartes' rule of signs, the number of positive roots of $Q(x)$ (resp., $Q(-x)$) is at most the number of sign changes in the list of coefficients 1, $1-\lambda$, $(-1)^n\lambda$ (resp. 1, $\lambda - 1$, $-\lambda$). The total of these sign changes is clearly ≤ 3. Noting $P(-x) = Q(-x)/(1-x)$ and so $x = 1$ counts as a positive root of $Q(-x)$ one more time than it does as a root of $P(-x)$, the above results for Q imply that P has at most two real roots.

Also solved by Con Amore Problem Group (Denmark), Robert Doucette, Michael Golomb, L. Van Hamme (Belgium), Thomas Jager, Eugene Levine, David E. Manes, Allan Pedersen (Denmark), Michael Vowe (Switzerland), A. Zulauf (New Zealand), and the proposers.

Pythagorean Triangle-Heronian Triangle Correspondence **February 1989**

1316. *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*

Characterize the Heronian triangles (a, b, c) in which the Eulerian segment OH between the circumcenter O and the orthocenter H subtends a right angle at the vertex A. (A *Heronian* triangle is one with integral sides and integral area.)

Solution by A. Zulauf, University of Waikato, Hamilton, New Zealand.

Let us say that (a, b, c) is a *Sastrian* triangle if and only if its sides a, b, c are integral and its Eulerian angle $\varepsilon = \angle HAO$ is a right angle and (u, v, w) is a *Pythagorean* triangle if and only if its sides u, v, w are integral and $u^2 + v^2 = w^2$. A Sastrian or Pythagorean triangle is *primitive* if and only if its sides have no common factor greater than 1.

Result. The mapping $a = |v^2 - u^2|$, $b = uw$, $c = vw$, is a one-to-one mapping from the set of all primitive Pythagorean triangles (u, v, w) onto the set of all primitive Sastrian triangles (a, b, c). All Sastrian triangles have integral area and thus are Heronian.

Proof. Suppose first that (a, b, c) is primitive Sastrian. Let $d = \gcd(b, c)$, $b = du$, $c = dv$. Let α, β, γ be the angles opposite a, b, c respectively. Since it is well known, and easy to show, that $|\varepsilon| = |\gamma - \beta|$, we deduce that $\gamma = \beta \pm \pi/2$, $\alpha = \pi - 2\beta \mp \pi/2$, and hence by the law of sines, that

$$\frac{a}{|\cos 2\beta|} = \frac{du}{\sin\beta} = \frac{dv}{|\cos\beta|},$$

whence

$$|\tan\beta| = \frac{u}{v}, \quad a = \left|\frac{\cos 2\beta}{\cos\beta}\right| dv = \frac{|1 - \tan^2\beta|}{\sqrt{1 + \tan^2\beta}}\, dv = \frac{|v^2 - u^2| d}{\sqrt{v^2 + u^2}}.$$

Since a is an integer, $u^2 + v^2$ must be a square integer, say w^2, so that (u, v, w) is primitive Pythagorean. Now w must be odd and relatively prime to u and v, so that

$$\gcd(w, v^2 - u^2) = \gcd(w, v^2 - u^2 + w^2) = \gcd(w, 2v^2) = 1.$$

It follows that w must be a divisor of d, say $d = kw$,

$$a = k|v^2 - u^2|, \qquad b = kuw, \qquad c = kvw,$$

and in fact $k = 1$, since $\gcd(a, b, c) = 1$. Moreover, the area of triangle (a, b, c) is given by

$$2 \times \text{area triangle}(a, b, c) = bc\sin\alpha = uvw^2|\cos 2\beta|$$
$$= uvw^2\left|\frac{1 - \tan^2\beta}{1 + \tan^2\beta}\right| = uv|v^2 - u^2|,$$

and thus the area is integral since uv must be even.

Conversely, suppose that (u, v, w) is primitive Pythagorean. Put $a = |v^2 - u^2|$, $b = uw$, $c = vw$. Then a, b and c are integers. Further, by the cosine and sine laws,

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{(v^2 - u^2)(v^2 - u^2 + w^2)}{2|v^2 - u^2|vw} = \pm\frac{v}{w},$$

$$\sin\gamma = \frac{c}{b}\sin\beta = \frac{v}{u}\sqrt{1 - \cos^2\beta} = \frac{v}{u}\sqrt{1 - \frac{v^2}{w^2}} = \frac{v}{w} = \pm\cos\beta.$$

This implies that $\gamma = \beta \pm \pi/2$, so that $\varepsilon = \pi/2$. Moreover,

$$\gcd(a, b, c) = \gcd(\gcd(b, c), a) = \gcd(w, v^2 - u^2) = 1.$$

Hence (a, b, c) is primitive Sastrian.

Finally, the mapping is one-to-one, since

$$|v^2 - u^2| = |V^2 - U^2|, \qquad uw = UW, \qquad vw = VW$$

is easily seen to imply that $u = U$, $v = V$, and $w = W$.

Also solved by J. C. Binz (Switzerland), Con Amore Problem Group (Denmark), Ragnar Dybvik (Norway), Volkhard Schindler (East Germany), Michael Vowe (Switzerland), and the proposer. There was one incomplete solution.

Answers

Solutions to the Quickies on p. 57.

A757. There is no such function. To see this, consider the cardinality of the image set. On the one hand, it is countable, because the image of the rationals is countable, and the image of the irrationals is countable because (in this case) it is a subset of the rationals. On the other hand, if a continuous function takes two different values then its image set contains an interval, and therefore is uncountable. So the function must be a constant function, but obviously this can't be the case.

A758. Letting $x = 2t$ in the first integral, we get

$$I = 2^{2n+1} \int_0^{a/2} t^n (a-t)^n \, dt \div \int_0^a t^n (a-t)^n \, dt.$$

Since

$$\int_0^a t^n (a-t)^n \, dt = \int_0^{a/2} t^n (a-t)^n \, dt + \int_{a/2}^a t^n (a-t)^n \, dt = 2 \int_0^{a/2} t^n (a-t)^n \, dt,$$

$I = 2^{2n}$.

This problem is due to E. B. Elliott and appears in Mathematical Problems from the *Educational Times*, where each integral is evaluated separately.

A759.

$$\begin{aligned}
&a^3b^3c^3 + b^3c^3d^3 + c^3d^3a^3 + d^3a^3b^3 \\
&= \frac{a^3b^3c^3 + b^3c^3d^3 + c^3d^3a^3}{3} + \frac{b^3c^3d^3 + c^3d^3a^3 + d^3a^3b^3}{3} \\
&\quad + \frac{c^3d^3a^3 + d^3a^3b^3 + a^3b^3c^3}{3} + \frac{d^3a^3b^3 + a^3b^3c^3 + b^3c^3d^3}{3} \\
&\geq \sqrt[3]{a^6b^6c^9d^6} + \sqrt[3]{b^6c^6d^9a^6} + \sqrt[3]{c^6d^6a^9b^6} + \sqrt[3]{d^6c^6b^9c^6} \\
&= a^2b^2c^3d^2 + b^2c^2d^3a^2 + c^2d^2a^3b^2 + d^2c^2b^3a^2 \\
&= a^2b^2c^2d^2(a + b + c + d) = a^2b^2c^2d^2\, P.
\end{aligned}$$

REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Waters, Tom, When reality doesn't reign, *Discover* (January 1990), 78.

Cipra, Barry, Hungarian mathematician squares the circle, *SIAM News* (September 1989), 19.

The Banach-Tarski paradox describes a decomposition of a solid ball into another with twice the radius, thus violating volume considerations by using nonmeasurable sets. However, equidecomposable figures in the plane or on the line must have the same area or length (because, as Banach showed, Lebesgue measure on either of those can be extended to a finitely additive measure defined on all subsets). Miklós Laczkovich (Eötvös Lorånd University, Budapest) has now shown that a disk can be finitely decomposed and rearranged into a square region of the same area, using only translations. The pieces have Jordan curves as boundaries.

Waters, Tom, Mathematics 1989: Pi in the sky, *Discover* (January 1990), 76-77.

At the limits of calculation: Pi to a billion digits or more, *Focus* 9:5 (October 1989), 1, 3-4.

Mathematics has had many notable achievements in 1989 (as noted in these reivews), but the one that captured the public imagination, the newspaper headlines, and the honor of one of the top ten science stories of 1989 (according to *Discover*) was the computation of pi to more than one billion decimal digits. The algorithm used comes from a Ramanujan-style series that arises in the theory of elliptic modular functions and is related to the quadratic field $Q(\sqrt{-163})$. Each excellent for its genre, the *Discover* article gives history of the computations of pi, while the *Focus* article gives references to mathematical literature. (Unfortunately, the author of the *Focus* piece is not identified.)

Peterson, Ivars, Natural selection for computers: Nature provides the model for a speedy computer search, *Science News* 136 (25 November 1989), 346-348.

Genetic algorithms, first explored by John Holland (Michigan) in the 1960s, are now coming to the fore. A genetic algorithm for a combinatorial optimization problem starts with a selection of feasible solutions. It then cycles through several "generations" of "crossing" pairs of solutions to produce new solutions, which are then evaluated and the "unfit" ones discarded. Current applications include designing minimum-weight trusses, aligning images in two different X-rays, running a gasline pumping station, and developing a strategy to pick winning horses at a racetrack.

Anderson, C.W., and R.M. Loynes, *The Teaching of Practical Statistics*, Wiley, 1987; xi + 199 pp, $59.95.

The most valuable part of this book on the aims and methods of teaching statistics is its collection of 36 short-term projects for students, together with discussion of the nature of open-ended projects. The author stresses the importance of teaching students how to write a report on a project.

Cipra, Barry, An astrophysical guide to the weather on earth, *Science* 246 (13 October 1989), 212-213.

A mathematical model originally devised for supernovas and solar flares now shows promise for applications in meteorology. Known as the *Piecewise Parabolic Method* (PPM), the model handles fluid flows possessing steep gradients, such as windshears and storm systems. The equations for fluid flow cannot be solved exactly, and the usual numerical techniques can't handle steep gradients. Instead of a constant or a linear approximation, PPM uses a separate parabola in each cell in the model space, hence its name.

Fonseca, James W., *Urban Rank-Size Hierarchy: A Mathematical Interpretation*, Institute of Mathematical Geography Monograph #8, Institute of Mathematical Geography (2790 Briarcliff, Ann Arbor, MI 48105), 1988; $15.95 (P).

Offers a fresh perspective on a continuing problem for geographers: how to explain the relationship betweent he relative sizes of cities and their size ranks. Various explanations have been offered in the past: lognormal, Pareto, and Yule distributions; logistic curve; and "rubbish." Fonseca offers a new descriptive explanation based on Fibonacci ideas and the equiangular spiral, with population plotted as radius and rank as angle.

Nicolis, Grégoire, and Ilya Prigogine, *Exploring Complexity: An Introduction*, Freeman, 1989; xi + 313 pp. ISBN 0-7167-1859-6

"What is the difference, if any, between the swinging of a pendulum; and the beating of the heart, or between a crystal of ice and a snowflake? Is the world of physical and chemical phenomena basically a simple and predictable world where all observed facts can be interpreted adequately by appealing to a few fundamental interactions? Is complexity to be found only in biology?" Self-organization in chemistry, dynamical systems, manifolds and fractals, attractors, chaos, morphogenesis, cellular automata: All these come into play, in this exploration by a Nobel-prize winner and his colleague. But it's not for the mathematically weak-at-heart: vector calculus and basic physics and chemistry are presumed.

Hirst, Ann and Keith Hirst, *Proceedings of the Sixth International Congress on Mathematical Education*, ICMI Secretariat and János Bolyai Mathematical Society (available from the MAA), 1989; 397 pp, $45 (from the MAA). ISBN 963-8022-48-5

Texts of the five plenary presentations at ICME 6 in Budapest in 1988, together with careful summaries of the discussions at the action, theme, and other groups.

Pool, Robert, Chaos theory: how big an advance?, *Science* 245 (7 July 1989), 26-28.

Is chaos "merely an interesting idea enjoying a faddish vogue ... or a revolution in scientific thought"? Will chaos achieve the status of quantum mechanics or relativity as a new scientific paradigm? This article, the last in a six-part series on how scientists are using chaos, lets exponents and detractors of chaos have their say.

Hazewinkel, M. (managing ed.), *Encyclopaedia of Mathematics*, Reidel, 1988- ; 10 vols., about 5000 pp, $149 per volume. ISBN 1-55608-010-7

An updated and annotated translation of the Soviet Mathematical Encyclopaedia (1977-1985). Includes three kinds of articles, most with references and editorial comments (updating): surveys; medium-length articles on more-detailed concrete problems, results, and techniques; and short definitions. One aim is completeness: "every theorem, concept, definition, lemma, construction which has a more-or-less constant and accepted name ... occurs somewhere, and can be found via the index." This is a work that your library definitely must have (and you probably should), never mind the price. The original Soviet version sold 150,000 copies (probably at one-twentieth the price). Anyone want to join a pool on how many this splendid English-language edition sells?

Dewdney, A.K., *The Armchair Universe: An Exploration of Computer Worlds*, Freeman, 1988; xiii + 330 pp, $19.95, $13.95 (P). ISBN 0-7167-1939-8

The first—of many such, I hope—collection of columns from Dewdney's "Computer Recreations" in *Scientific American*, with addenda giving updates and reader contributions. Readers are offered the opportunity both to explore by programming and to survey the questions that the programs' results provoke: the Mandelbrot set, wallpaper for the mind, hypercubes, perceptron misperceptions, busy beavers, the evolution of fibs, ...

Ralston, Anthony (ed.), *Discrete Mathematics in the First Two Years*, MAA, 1989; 101 pp, $10(P). ISBN 0-88385-064-8

Recounts the experiences of six colleges and universities funded by the Sloan Foundation in 1984-86 to experiment with integration of discrete and continuous mathematics in the first two years of college mathematics. Includes course examinations.

Barbeau, E.J., *Polynomials*, Springer-Verlag, 1989; xxii + 441 pp, $59. ISBN 0-387-96919-5

In an era in which the curriculum of high-school mathematics has been bowdlerized, here is a book to "encourage students to dwell on a mathematical topic long enough to sense how it is put together and what its proper context is." The subject is theory of equations, and the emphasis is on depth and breadth. The vehicles are exercises (to introduce the basic ideas and theory through examples), explorations (to raise questions and encourage investigation), and problems (from the *American Mathematical Monthly*, *Crux Mathematicorum*, Olympiads, etc.). The experience involved may be of far greater mathematical value to a high-school senior than galloping through the mechanics of calculus.

National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards for School Mathematics*, NCTM, 1989; vii + 258 pp, $25.

Growing out of the various commission studies of the past five to ten years, this report attempts to prescribe specifics of content and skills for each grade K-12. Mathematics teachers will be paying attention to this work, and teacher candidates (as well as their professors) should become familiar with it. The next steps will be to make concrete the specifics into a curriculum of lessons and tasks, through writing texts and training teachers. But changing the curriculum's thrust or even some specifics will not likely change the 40-70% failure rates in Algebra I in city schools, nor address what many teachers regard as their real problems: apathy, discipline, and absenteeism ("activity" absences as well as truancy).

Manber, Udi, *Introduction to Algorithms: A Creative Approach*, Addison-Wesley, 1989; xiv + 478 pp.

"This book emphasizes the creative side of algorithm design. Its main purpose is to show the reader how to design a new algorithm." Intended for a course on algorithms, this thrilling book emphasizes the central role of mathematical induction in designing and refining algorithms.

Duren, Peter, *A Century of Mathematics in America, Part III*, American Mathematical Society, 1989; ix + 675 pp. ISBN 0-8218-0136-8

Third and final volume of a project of newly-written and reprinted historical articles marking the centennial of the American Mathematical Society. Part III, like Part II, contains historical articles on departments of mathematics at leading American universities: Johns Hopkins, Clark, Columbia, MIT, Michigan, Texas, and the Institute for Advanced Study (institutions were selected based on willingness of qualified people to write about them). Also included are some individual biographies; surveys on various topics; a section on probability, statistics, and actuarial science; and a couple of articles on sources.

Mathews, Jay, *Escalante: The Best Teacher in America*, Henry Holt & Co., 1988; ix + 322 pp, $9.95 (P). ISBN 0-8050-1195-1

Journalist's account of the man, the school, and the success in calculus documented in the film *Stand and Deliver*. (I can see Escalante now: "Just when I was beginning to live down the film and get back to teaching as usual, here comes this book with its embarrassing subtitle and my picture on the cover ..."). If we're lucky, maybe it will be featured at supermarket checkstands.

Cox, David A., *Primes of the Form $x^2 + ny^2$: Fermat, Class Field Theory, and Complex Multiplication*, Wiley, 1989; xi + 351 pp, $42.95.

"Several years ago while reading Weil's *Number Theory* ..., I noticed a conjecture of Euler concerning primes of the form $x^2 + 14y^2$. That same week I picked up Cohn's *A Classical Invitation to Algebraic Numbers and Class Fields* and saw the same example treated from the point of view of the Hilbert class field. The coincidence made it clear that something interesting was going on, and this book is my attempt to tell the story of this wonderful part of mathematics." This is a book at the second-year graduate level, with a beautiful unity and a compelling use of a variety of mathematical tools.

The Mathematics and Knots Exhibition Group, *Mathematics and Knots*, University of Wales at Bangor; 27 pp (P). ISBN 0-9514947-1-6

Booklet based on the exhibition "Mathematics and Knots" designed in the School of Mathematics at the University of Wales at Bangor. Exposits the topology and arithmetic of knots, in an easy-to-understand fashion.

Morel-Deledalle, Myriame, et al., *Mathématiques en Méditerranée: des tablettes babyloniennes au théorème de Fermat*, Édisud/Musées de Marseille, 1988; 120 pp, 100 Frs. ISBN 2-85744-365-X

Pardon me for reviewing a beautiful book that your bookseller or librarian is going to have to work hard if you are ever going to see a copy—not to mention that the book is in a foreigh language! It is the lavishly-illustrated companion to an exhibit at the Museums of Marseille, which sprang from a 1984 colloquium in Marseille on the mathematics of the Mediterranean.

American Association for the Advancement of Science, *Science for All Americans: A Project* 2061 *Report on Literacy Goals in Science, Mathematics, and Technology*, AAAS, 1989; ix + 217 pp, \$14.50 (P). ISBN 0-87168-341-5. Blackwell, David, and Leon Henkin, *Mathematics: Report of the Project* 2061 *Phase I Mathematics Panel*, AAAS, 1989; xi + 47 pp, \$7.50 (P). ISBN 0-87168-344-X.

Halley's comet will appear again in 2061, and the children who will live to see that return will soon be starting school. What should be the substance and character of their education for scientific literacy? This prestigious report gives answers concerning the nature of the scientific endeavor, basic knowledge of the world, what people should understand about some of the great episodes in science, and habits of mind that are essential for scientific literacy. Ideas and thinking skills are emphasized at the expense of memorization of details (terminology, formulas, and algorithms). The subpanel report on mathematics asks, "What are the important ideas of mathematics that people should know and understand by the age of 18?" (there are no surprises in the answers). Companion reports treat technology, social and behavioral sciences, biological and health sciences, and physical and information sciences and engineering. Phase I of Project 2061 focussed on the substance of scientific literacy; Phase II will present several alternative curriculum models; and Phase III (lasting a decade or longer) will involve implementation.

Chern, S. S. (ed.), *Global Differential Geometry*, MAA Studies in Mathematics Vol. 27, MAA, 1989; ii + 354 pp, \$36.50. ISBN 0-88385-129-6

Essentially a second edition of MAA Studies Vol. 4, *Global Geometry and Analysis* (1967). Most of the original chapters have been updated, and four new chapters cover Riemannian geometry of manifolds with nonnegative and with nonpositive curvature, the 1986 result on the Gauss map of a complete minimal surface in R^3 which is not a plane (the map can omit at most 4 points), and vector bundles with a connection.

Swann, Howard, and John Johnson, *Prof. E. McSquared's Calculus Primer*, Expanded Intergalactic Version, Janson Publications, 1989; 261 pp, \$15.50 (P). ISBN 0-939765-8

The famous "cartoon calculus" book, in an expanded version: new extra exercises and excursions, and a new Intergalactic Epilogue. Who says learning calculus can't be fun too?

Puccia, Charles J., and Richard Levins, *Qualitative Modeling of Complex Systems: An Introduction to Loop Analysis and Time Averaging*, Harvard U Pr, 1985; viii + 259 pp. ISBN 0-674-74110-2

"Loop models" is another name for signed digraphs, which provide a powerful tool for modeling interacting phenomena in qualitative terms. This book provides a thorough treatment; for less-extensive treatments, see F. S. Roberts and T. A. Brown, "Signed digraphs and the energy crisis," *Amer. Math. Monthly* 82 (1975), 577-594, and F. S. Roberts, *Discrete Mathematical Models, with Applications to Social, Biological, and Environmental Problems*, Prentice Hall, 1976, pp. 176-257.

NEWS AND LETTERS

LETTERS TO THE EDITOR

Editor:

Soon after my article (this MAGAZINE 62 (1989), 35-37) on a geometrical approach to Cramer's Rule appeared I obtained a copy of Grassmann's *Ausdehnungslehre* in its two versions of 1844 (A 1) and 1861/2 (A 2), to which my attention had been drawn by a reference in Michael J. Crowe's *History of Vector Analysis* (Dover, NY, 1985). I find that much of what I wrote is implicit in Grassmann's work (see especially §§ 45, 46 and 92 of A 1; Appendix III, 12, to the 1878 reprint of A 1; and Numbers 62, 63, 134 and 377 of A2, with the note to Nr. 134 by Hermann Grassmann Junior in Grassmann's collected *Mathematical and Physical Writings*, Volume I, part II, Teubner, Leipzig, 1896). His derivation of Cramer's Rule is readily given a geometrical interpretation, and if so, yields in addition a rather simple proof that determinants may be regarded as volumes. Grassmann, however, saw his Extension Theory (Ausdehnungslehre) as forming the abstract basis of geometry, freed from all spatial ideas (Appendix III, I), and it was left to his son to provide such an interpretation, in the note referred to above.

Perhaps I may also draw attention here to Grassmann's rule for the sign of products (Nr. 62 of A 2), which is claimed by his Editor to be equivalent to Cramer's, but which appears to break down for more than three dimensions.

J. W. Orr
Crookstown
County Cork, Ireland

* * * * *

Editor:

I would like to comment on a recent paper in *Mathematics Magazine* by L. Hoehn and J. Ridenhour [3].

The authors' Theorem 1 was discovered and proved in 1849 by G. L. Dirichlet [1]. Dirichlet considered the function

$$D(x) = \sum_{1\leq n\leq x} d(n)$$

where $d(n)$ is the number of divisors of n, and proved that

$$D(x) = \sum_{1\leq n\leq x} \left[\frac{x}{n}\right] = 2\sum_{1\leq n\leq\sqrt{x}} \left[\frac{x}{n}\right] - [\sqrt{x}]^2$$

for $x \geq 1$. He also considered the function

$$\Delta(x) = D(x) - x\log x - (2\gamma - 1)x$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant, and demonstrated in an elementary way that $\Delta(x) = O(\sqrt{x})$ as $x \to \infty$. The authors of [3] use the notation $S(x) = D(x)$ and prove that $\Delta(x) = o(x)$ as $x \to \infty$.

In 1903 in a remarkable paper G. Voronoi [7] demonstrated that

$$\left|\Delta(x) - \frac{1}{4}\right| < \frac{65}{36}\, x^{1/3}\log x + \frac{79}{12}\, x^{1/3} + \frac{3}{2}$$

for $x \geq 1$. Voronoi's proof is lengthy, but elementary. In 1915 E. Landau [5] and G. H. Hardy [2] proved that if $\Delta(x) = O(x^{\alpha})$, then necessarily $\alpha \geq 1/4$. In 1922 J. G. van der Corput [6] proved that $\Delta(x) = O(x^{\alpha})$ for some $\alpha < 0.33$. This result has been improved by later researches. In 1969 G. A. Kolesnik [4] proved that $\Delta(x) = O(x^{12/37}(\log x)^{62/37})$ as $x \to \infty$.

Note that in Table 3 of [3], $n(\log n + 2\gamma - 1)$ is correctly 162,725,2<u>70</u> for $n = 10^7$ and 1,857,511,<u>207</u> for $n = 10^8$.

Finally, I would like to note that in Table 4 of [3], $S(n)n^{-1} + 1 - \gamma$ is not a good approximation of

$$H(n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$$

because the function $S(n)$ oscillates wildly, and the error of the approximation is at least $O(n^{-3/4})$. On the other hand

$$0 < H(n) - \log n - \gamma < \frac{1}{2n}$$

for $n \geq 1$, that is, $\log n + \gamma$ is a much better approximation of $H(n)$.

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2. G. H. Hardy, On Dirichlet's divisor problem, *Proc. London Math. Soc.* 15 (1916), 1-25.

3. L. Hoehn and J. Ridenhour, Summations involving computer-related functions, this MAGAZINE 62 (1989), 191-196.

4. G. A. Kolesnik, Improvement of remainder term for the divisor problem, *Math. Notes* 6 (1969), 784-791.

5. E. Landau, Über Dirichlet's Teilerproblem, *Sitzungsber. Math.-Phys. Kl. Königlich Bayer. Akad. Wiss.*, 1915, pp. 317-328.

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7. G. Voronoi, Sur un problème du calcul des fonctions asymptotiques, *J. Reine Angew. Math.* 126 (1903), 241-282.

L. Takács
Case Western Reserve University
Cleveland, OH 44106

Editor:

We appreciate the historical background of the Dirichlet problem provided by Professor Takács. Our purpose in writing the article was not to present an historical or research article, but rather an expository one accessible to undergraduates showing the interrelation of mathematics and computer science. Rather than calculating sums with extremely large numbers of terms, mathematical analysis often allows us to calculate equivalent sums with far fewer terms. In particular, our Theorem 2 reduced a sum of 10,000,000 terms to one of ten terms when $r = 7$ (the value for r was omitted in the article).

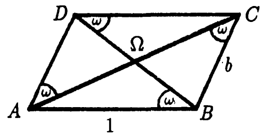
Larry Hoehn
Jim Ridenhour
Austin Peay State University
Clarksville, TN 37044

* * * * *

Editor:

A parallelogram is called *self-diagonal* if its sides are proportional to the diagonals. Let $ABCD$ be a parallelogram, $AB = a$, $BC = b$, $B\hat{A}C \leq 90^\circ$, $a \geq b$. Then the self-diagonality is easily seem to be equivalent to $AC = \sqrt{2}a$, $BD = \sqrt{2}b$. The triangle inequality $AB + BC \geq AC$ and the cosine law determine S-D parallelograms $a = 1$, $b \in \left[\sqrt{2} - 1, 1\right]$, $B\hat{A}C = \cos^{-1}\left(\frac{1 - b^2}{2b}\right)$ completely up to similarity. When $B\hat{A}C = 60^\circ$ then b is the golden ratio and hence *self-diagonal golden parallelograms* are similar to $\left(1, \frac{1}{2}(\sqrt{5} - 1), 60^\circ\right)$ parallelograms.

Now only S-D-Ps admit an extension of Brocard geometry for triangles. In the figure, $ABCD$ is S-D-P.



The similarity of the triangles $A\Omega D$ and ABD yields the Brocard angle $\omega = D\hat{A}\Omega = A\hat{B}\Omega = B\hat{C}\Omega = C\hat{D}\Omega$. Also, the Brocard point Ω is the intersection of four (Brocard) circles such as $A\Omega B$ tangent to DA at A. The converse that if a parallelogram has Brocard angle ω or Brocard point Ω then it must be S-D-P follows. Sin $A = \sqrt{2}$ sin ω yields $0^\circ \leq \omega \leq 45^\circ$. As b moves from $(\sqrt{2} - 1)$ to 1, the locus of Ω is the quadrant of circle with center at A and radius $\sqrt{2}/2$. The referee has kindly provided three references to the literature.

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4. K. Hagge, Die Grundlagen der Brocardschen Geometrie des Dreiecks und die Erweiterung auf das Vieleck, *Jahresber. Deutsch. Math.-Verein.* 39, Heft 1/4 (1930), 81-88.

K. R. S. Sastry
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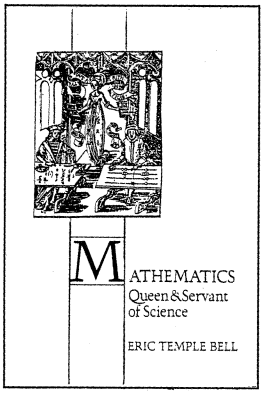

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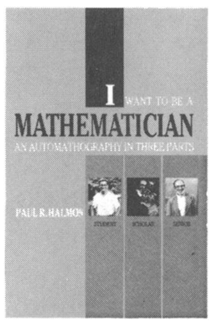

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THE USE OF CALCULATORS IN THE STANDARDIZED TESTING OF MATHEMATICS

John W. Kenelly, Editor

The calculator is a universal tool for all those involved in quantitative work from science and engineering to business. Routine use of calculators is part of the training and testing of students headed for these fields. But this is not yet the case in mathematics. This symposium, jointly sponsored by the MAA and The College Board, sets out clearly the theoretical and practical issues that must be addressed as calculators are brought more fully into the mathematics curriculum. This is a practical group concerned with specific tests and test items. General theoretical considerations are set off by the specifics of individual test items and students' success rates on them. The Ohio Early College Mathematics Placement Test is reported on in detail by Joan R. Leitzel and Bert K. Waits. James W. Wilson and Jeremy Kilpatrick examine the theoretical issues in the development of calculator-based tests. John Harvey, now Chair of the MAA's Committee on Placement Examinations looks at the issues surrounding calculator use on placement examinations, as well as giving an overview of the symposium and a survey of developments through 1988.

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